MINIMUM-FUEL FORMATION RECONFIGURATION OF MULTIPLE FREE-FLYING SPACECRAFT

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Abstract

The formation reconfiguration of multiple free-flying spacecraft is posed as an optimization problem involving minimum fuel expenditure. A method for obtaining simple solutions to this problem is proposed. The basic idea is to break up the formation reconfiguration process into a sequence of simple manuevers involving a small number of spacecraft at a time. Moreover, the spacecraft move along straightline paths in the 3-dimensional space. Thus, the minimum-fuel reconfiguration problem can be reduced to an optimization problem involving permutation groups. The proposed approach requires seeking fuel minimization over the set of all spacecraft permutation cycles of lengths ≥ 2. Examples are given to illustrate the application of the proposed method.

Introduction

Recently, the use of multiple spacecraft for long base-line interferometers, magnetosphere studies, and space-based communication networks was considered. Multiple-spacecraft interferometry was first proposed by Stachnik and his coworkers [1],[2]. Its feasibility for LEO (Low-Earth-Orbit) was studied by DeCue [3] with emphasis on using thrust-free orbits to reduce the amount of necessary control effort for formation keeping. A description of the recently proposed New Millennium separated spacecraft interferometer was given in [4]. Subsequently, various problems associated with the coordination and control of formation flying of multiple spacecraft were studied. Wang and Hadaegh [5],[6] derived various control

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laws for the coordination and control of free-flying and LEO multiple spacecraft moving in formation using a Lyapunov approach. No optimization criteria were considered. A collection of papers and presentations on formation flying up to 1996 was complied by Lau [7]. Recently, various key issues associated with formation flying were identified in a workshop on formation flying and micro-propulsion [8]. It is obvious that efficient use of fuel is one of the main issues in any spacecraft maneuvers. In physical situations, failure in one or more spacecraft may occur, it is necessary to consider different options in maintaining the spacecraft formation without impairing its mission. The failure may take on various forms. When the failure is sufficiently severe such that the spacecraft is no longer useful, removal of the spacecraft from the formation is necessary. A possible option is to reconfigure the formation. Also, a change of mission objectives during flight may require formation reconfiguration. For example, in the case of an interferometer, formation reconfiguration may be required to suit a new target. Since the spacecraft have limited fuel onboard, it is important that the formation reconfiguration manuevers are accomplished with minimum fuel expenditure. Beard, McLain and Hadaegh [9] considered the problem for equalizing the fuel expenditure for multiple spacecraft during a retargeting maneuver without formation reconfiguration.

In this paper, we present a method for obtaining *simple* solutions to the minimum-fuel formation reconfiguration problem. The basic idea is to break up the formation reconfiguration process into a sequence of simple manuevers involving a small number of spacecraft. In the development of this paper, the formulation of the formation reconfiguration problem is discussed first. Then an analysis of the basic manuevers which are relevant to the proposed solution method is presented. The application of the proposed method for solving two types of minimum-fuel formation reconfiguration problems is discussed in detail, and illustrated by examples.

Spacecraft Formation

In general, a formation may be composed of identical spacecraft as in the ESA/NASA Cluster [10], or not all identical spacecraft as in the proposed NASA DS-3 interferometer [4]. In the latter case, the combiner and collector spacecraft have different structures and characteristics. In this study, we consider a spacecraft formation consisting of P subsets of free-flying spacecraft in the absence of gravitational fields or disturbances. Each subset $S_j \subset \mathcal{I}$ consists of N_j *identical* spacecraft, and $\sum_{j=1}^{P} N_j = N$, where $\mathcal{I}$ denotes the index set $\{1, \ldots, N\}$.

Let $\mathcal{F}_o$ denote the inertial frame with origin O in the 3-dimensional Euclidean space R^3. Given a basis $\mathcal{B}_o = \{e_1, e_2, e_3\}$ for $\mathcal{F}_o$, the representation of a vector $\mathbf{a} = \sum_{i=1}^{3} a_i e_i \in R^3$ with respect to $\mathcal{B}_o$ is denoted by the column vector $[\mathbf{a}]_o = (a_1, a_2, a_3)^T$. Let $\mathbf{r}_i(t)$ and $\mathbf{r}_i^d(t)$ denote respectively the actual and desired positions of the mass center of the i-th spacecraft at time t relative to $\mathcal{F}_o$. The point set $\mathcal{P}(t) = \{\mathbf{r}_1(t), \ldots, \mathbf{r}_N(t)\}$ generates a *formation pattern* at time t. The convex polytope $\mathcal{C}(t)$ defined by $co(\mathcal{P}(t))$ (the convex hull of $\mathcal{P}(t)$) is referred to hereafter as the *formation body* at time t. In formation reconfiguration, a *desired formation pattern* given by $\mathcal{P}^d(t) = \{\mathbf{d}_1(t), \ldots, \mathbf{d}_N(t)\}$ defined for each t in some time interval I_T is specified, where $\mathbf{d}_i(t)$ corresponds to the position of the i-th point in the desired formation pattern at time t relative to $\mathcal{F}_o$. The formation patterns $\mathcal{P}(t)$ (or $\mathcal{P}^d(t)$), $t \in I_T$, are said to be *shape-invariant* over some time interval I_T, if the Euclidean distance between any pair of distinct points in the formation pattern at time t is constant for all $t \in I_T$. This implies that the geometric shape of the formation body does not vary with time over I_T.

In many physical situations involving subsets S_j of identical spacecraft, it is only required that each point in the desired formation point set $\mathcal{P}^d(t)$ be occupied by some spacecraft from a specified S_j. Thus, the i-th element $\mathbf{d}_i(t)$ in $\mathcal{P}^d(t)$ may not correspond to $\mathbf{r}_i^d(t)$. For

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example, for a spacecraft triad in a triangular formation with $S_1 = \{1\}, S_2 = \{2,3\}$, and $\mathcal{P}^d(t) = \{\mathbf{d}_1(t), \mathbf{d}_2(t), \mathbf{d}_3(t)\}$, we only require $\mathbf{d}_1(t) = \mathbf{r}_1^d(t), (\mathbf{d}_2(t), \mathbf{d}_3(t)) = (\mathbf{r}_2^d(t), \mathbf{r}_3^d(t))$ or $(\mathbf{r}_3^d(t), \mathbf{r}_2^d(t))$.

In formation design, one may wish to include a number of spare spacecraft in the formation for backup in the event of spacecraft failure. The spare spacecraft type should be selected by considering their importance in maintaining the basic functions of the formation. The spare spacecraft position relative to the formation should be chosen so that they do not interfere with formation task performance, and also permits efficient formation reconfiguration in case of spacecraft failure. For example, for a regular tetrahedral formation as in the ESA/NASA Cluster, it may be desirable to have the spare spacecraft located near the center of the tetrahedron so that they are nearly equidistant to any active spacecraft.

Formation Reconfiguration

Formation reconfiguration can be classified into two basic types. In Type 1, each spacecraft is required to occupy a specified position in the desired reconfigured formation, while as in Type 2, a specified position in the desired reconfigured formation may be occupied by *any* spacecraft of a particular type. In general, the total number of spacecraft before and after reconfiguration may differ due to the presence of failed spacecraft, and/or augmentation of the number of spacecraft to attain a larger formation. This situation can also arise when a large formation is partitioned into a number of small formations, or enlarged by merging a collection of small formations.

In the case where the total number of spacecraft is less than that of the initial formation, it is possible that only a limited number of useful formation patterns can be generated from the reduced number of spacecraft, especially when the spacecraft are not all identical. In fact, it may be necessary to remove additional spacecraft before a useful geometric formation pattern

can be obtained. For example, consider an initial formation consisting of two different types of equally numbered spacecraft placed on a circle with equal spacing. In order to retain the circle formation with equal spacecraft spacing after reconfiguration, the number of these two types of spacecraft must be the same. Thus, any removal must be made in pairs of nonidentical spacecraft. In what follows, we assume that the total number of spacecraft before the initiation of the reconfiguration process has been adjusted to coincide with that in the desired reconfigured formation.

In the formation reconfiguration manuevers, it is important to minimize the total fuel expenditure. In the absence of gravitational field and other disturbances, this can be achieved by requiring the spacecraft to move along straightline paths in R^3. Thus, a basic spacecraft manuever is to move from one specified point to another along a straightline path with a minimum amount of fuel. In what follows, the controls for this type of manuevers will be discussed first. Then, the minimum-fuel formation reconfiguration problem for each type of formation reconfiguration will be discussed separately.

Basic Manuevers Consider an initial formation whose pattern at time $t \geq 0$ is given by $\mathcal{P}^o(t) = \{\mathbf{r}_1^o(t), \ldots, \mathbf{r}_N^o(t)\}$. The evolution of $\mathbf{r}_i^o(t)$ with time t is governed by

$$\frac{d^2 \mathbf{r}_i^o(t)}{dt^2} = \frac{\mathbf{f}_{ci}^o(t)}{M_i}, \qquad i = 1, \ldots, N, \tag{1}$$

where $\mathbf{f}_{ci}^o = \mathbf{f}_{ci}^o(t)$ is a given thrust program for generating $\mathbf{r}_i^o = \mathbf{r}_i^o(t)$, and M_i is the mass of the i-th spacecraft.

Let the position of the i-th spacecraft in the initial formation with respect to the inertial frame $\mathcal{F}_o$ at time t be given by $\mathbf{r}_i^o(t)$. Let

$$\rho_{di}^o(t) = \mathbf{r}_i^d(t) - \mathbf{r}_i^o(t), \tag{2}$$

and

$$\mathbf{w}_i(t) = \rho_{di}^o(t) / \|\rho_{di}^o(t)\|, \tag{3}$$

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where $\| \cdot \|$ denotes the usual Euclidean norm.

A basic manuever is to translate the i-th spacecraft in the direction $\mathbf{w}_i(t)$ to its target position $\mathbf{r}_i^d(T_i)$ at some time $T_i > 0$. Let the actual and desired positions $\mathbf{r}_i(t)$ and $\mathbf{r}_i^d(t)$ of the i-th spacecraft during translation be described respectively by

$$\frac{d^2\mathbf{r}_i(t)}{dt^2} = \frac{\mathbf{f}_{ci}(t)}{M_i}, \qquad \frac{d^2\mathbf{r}_i^d(t)}{dt^2} = \frac{\mathbf{f}_i^d(t)}{M_i}, \tag{4}$$

where $\mathbf{f}_{ci}$ denotes the actual control thrust for the i-th spacecraft, and $\mathbf{f}_i^d = \mathbf{f}_i^d(t)$ is the required thrust program for generating $\mathbf{r}_i^d = \mathbf{r}_i^d(t)$.

Consider the projection of $(\mathbf{r}_i^d(t) - \mathbf{r}_i(t))$ onto the direction $\mathbf{w}_i(t)$ given by

$$s_i(t) = \langle \mathbf{r}_i^d(t) - \mathbf{r}_i(t), \mathbf{w}_i(t) \rangle, \tag{5}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on R^3. By differentiation and making use of (1) and (4), we have

$$\frac{d^2 s_i(t)}{dt^2} = \left\langle \frac{\mathbf{f}_i^d(t) - \mathbf{f}_{ci}(t)}{M_i}, \mathbf{w}_i(t) \right\rangle + 2 \left\langle \frac{d(\mathbf{r}_i^d(t) - \mathbf{r}_i(t))}{dt}, \frac{d\mathbf{w}_i(t)}{dt} \right\rangle + \left\langle \mathbf{r}_i^d(t) - \mathbf{r}_i(t), \frac{d^2\mathbf{w}_i(t)}{dt^2} \right\rangle. \tag{6}$$

We assume that $\mathbf{r}_i^o(t)$ and $\mathbf{r}_i^d(t)$ have the forms:

$$\mathbf{r}_i^o(t) = \mathbf{r}_i^o(0) + \mathbf{q}(t), \qquad \mathbf{r}_i^d(t) = \mathbf{r}_i^d(0) + \mathbf{q}(t), \tag{7}$$

where $\mathbf{q} = \mathbf{q}(t)$ is a given function of t, representing the formation drift vector. Then, $\mathbf{w}_i(t) = (\mathbf{r}_i^d(0) - \mathbf{r}_i^o(0))/\|\mathbf{r}_i^d(0) - \mathbf{r}_i^o(0)\|$ is a constant vector, implying that the distance between $\mathbf{r}_i^d(t)$ and $\mathbf{r}_i^o(t)$ is time invariant, and $\mathbf{f}_{ci}^d(t) = \mathbf{f}_{ci}^o(t)$. Thus, (6) reduces to

$$\frac{d^2 s_i(t)}{dt^2} = \left\langle \frac{\mathbf{f}_i^d(t) - \mathbf{f}_{ci}(t)}{M_i}, \mathbf{w}_i(t) \right\rangle \stackrel{\text{def}}{=} \frac{\hat{u}_i(t)}{M_i}, \tag{8}$$

where $\hat{u}_i$ corresponds to the effective control along the direction $\mathbf{w}_i(t)$. We assume that the admissible $\hat{u}_i = \hat{u}_i(t)$ are piecewise continuous functions of t satisfying $|\hat{u}_i(t)| \le \bar{u}_i$ for all t,

where $\bar{u}_i$ is a specified positive constant. Let $\tilde{s}_i = s_i M_i / \bar{u}_i$ and $u_i = \hat{u}_i / \bar{u}_i$. Then, (8) can be rewritten in the following normalized form:

$$\frac{d^2 \tilde{s}_i}{dt^2} = u_i. \tag{8'}$$

Now, given a terminal time $T_i < \infty$, the minimum-fuel translational manuever problem is to find an admissible normalized control $u_i^* = u_i^*(t)$ defined on the time interval $[0, T_i]$ which steers the initial state $\tilde{s}_i(0) = s_i(0) M_i / \bar{u}_i = \| \rho_{di}^o(0) \| M_i / \bar{u}_i, (d\tilde{s}_i/dt)(0) = 0$ to the target state at time T_i given by $\tilde{s}_i(T_i) = 0, (d\tilde{s}_i/dt)(T_i) = 0$ such that the fuel expenditure associated with the translational manuever given by

$$F(\dot{u}_i) = \alpha_i \int_0^{T_i} |u_i(t)| dt \tag{9}$$

is minimized, where α_i is a specified positive proportionality constant. Here, we have assumed that the rate of fuel consumption at any time is proportional to the magnitude of the control variable at time t.

The complete solution to this problem was given in [10]. Let $x_1 = \tilde{s}_i$, and $x_2 = dx_1/dt$ so that (8') takes on the standard form:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ u_i \end{bmatrix}, \tag{10}$$

and the optimal control problem corresponds to finding an admissible control defined on the time interval $[0, T_i]$ such that the initial state $\mathbf{x}_o = (x_{1o}, x_{2o})^T$ is steered to the zero state at time T_i with minimum fuel expenditure.

Let

$$\gamma_+ = \{(x_1, x_2) \in R_2 : x_1 = \frac{1}{2} x_2^2, x_2 \leq 0\}; \qquad \gamma_- = \{(x_1, x_2) \in R_2 : -\frac{1}{2} x_2^2, x_2 \geq 0\}; \tag{11}$$

$$\mathcal{R}_1 = \{(x_1, x_2) \in R_2 : x_2 \geq 0; x_1 > x_1', \text{where } (x_1', x_2) \in \gamma_-\};$$

$$\mathcal{R}_2 = \{(x_1, x_2) \in R_2 : x_2 > 0; x_1 < x_1', \text{where } (x_1', x_2) \in \gamma_-\};$$

$$\mathcal{R}_3 = \{(x_1, x_2) \in R_2 : x_2 \leq 0; x_1 < x_1', \text{where } (x_1', x_2) \in \gamma_+\};$$

$$\mathcal{R}_4 = \{(x_1, x_2) \in R_2 : x_2 < 0; x_1 > x_1', \text{where } (x_1', x_2) \in \gamma_+\}. \tag{12}$$

The above sets are shown in Fig.1.

Given an initial state $\mathbf{x}_o = (x_{1o}, x_{2o})^T \in R_2$ at time $t = 0$, the foregoing problem has a solution if and only if $T_i \geq T_i^*$, where T_i^* is the minimum time for steering $\mathbf{x}_o$ to the zero state, or

$$T_i \geq T_i^* = \begin{cases} x_{2o} + \sqrt{4x_{1o} + 2x_{2o}^2} & \text{for } \mathbf{x}_o \in \mathcal{R}_1 \cup \mathcal{R}_4, \\ -x_{2o} + \sqrt{-4x_{1o} + 2x_{2o}^2} & \text{for } \mathbf{x}_o \in \mathcal{R}_2 \cup \mathcal{R}_3, \\ |x_{2o}| & \text{for } \mathbf{x}_o \in \gamma_+ \cup \gamma_-. \end{cases} \tag{13}$$

Assuming that condition (13) is satisfied, then for all $\mathbf{x}_o \in \mathcal{R}_1$ and all $0 < T_i < \infty$; or $\mathbf{x}_o \in \mathcal{R}_4$ and all $T_i \leq -\frac{1}{2}x_{2o} - \frac{x_{1o}}{x_{2o}}$, the optimal control $u^* = u^*(t)$ is given uniquely by

$$u^*(t) = \begin{cases} -1 & \text{for } 0 \leq t < t_1, \\ 0 & \text{for } t_1 \leq t < t_2, \\ 1 & \text{for } t_2 \leq t \leq T_i. \end{cases} \tag{14}$$

where

$$t_1 = \frac{1}{2}\left\{ T_i + x_{2o} - \sqrt{(T_i - x_{2o})^2 - 4x_{1o} - 2x_{2o}^2} \right\},$$

$$t_2 = \frac{1}{2}\left\{ T_i + x_{2o} + \sqrt{(T_i - x_{2o})^2 - 4x_{1o} - 2x_{2o}^2} \right\}. \tag{15}$$

The corresponding minimum fuel expenditure is given by

$$F(u_i^*) = \alpha_i \left\{ T_i - \sqrt{(T_i - x_{2o})^2 - 4x_{1o} + 2x_{2o}^2} \right\}. \tag{16}$$

A realization of the foregoing optimal control in feedback form is given by

$$u^*(x_1, x_2) = -\text{Sgn}(x_1 + \frac{1}{2}x_2|x_2|) - \text{Sgn}(x_1 + m_\beta x_2|x_2|), \tag{17}$$

where

$$m_\beta = \frac{\beta}{2\beta - 2\sqrt{\beta(\beta - 1)} - 1} - \frac{1}{2}, \qquad \beta = T_i/T_i^*. \tag{18}$$

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The foregoing optimal control is relevant to the minimum-fuel formation reconfiguration problem to be considered later. The optimal controls corresponding to other conditions on $\mathbf{x}_o$ and T_i are given in [11]. It can be seen from (14) that an optimal trajectory consists of the trajectory due to an initial full thrust in the direction of $\mathbf{w}_i(t)$, a coasting trajectory, and a trajectory due to full thrust opposite the direction of $\mathbf{w}_i(t)$ (see Fig.1).

Another basic manuever is to transpose or interchange the positions of a spacecraft pair, say the i-th and j-th spacecraft. In this case, we set

$$\mathbf{r}_i^d(t) = \mathbf{r}_j^o(t), \qquad \mathbf{r}_j^d(t) = \mathbf{r}_i^o(t), \tag{19}$$

and define

$$\rho_{di}^o(t) = \mathbf{r}_j^o(t) - \mathbf{r}_i^o(t), \qquad \mathbf{w}_i(t) = \rho_{di}^o(t)/\|\rho_{di}^o(t)\|, \qquad \mathbf{w}_j(t) = -\mathbf{w}_i(t). \tag{20}$$

As before, we consider the projection of $(\mathbf{r}_i(t) - \mathbf{r}_i^o(t))$ onto the direction $\mathbf{w}_i(t)$ given by (5). Under the assumption that

$$\mathbf{r}_i^o(t) = \mathbf{r}_i^o(0) + \mathbf{q}(t), \qquad \mathbf{r}_j^o(t) = \mathbf{r}_j^o(0) + \mathbf{q}(t), \tag{21}$$

where $\mathbf{q} = \mathbf{q}(t)$ is the formation drift vector as in (7), then $\mathbf{w}_i(t) = \mathbf{r}_j^o(0) - \mathbf{r}_i^o(0)$ is a constant vector, implying that the initial or desired formation patterns for $t \geq 0$ are shape-invariant. Thus, Eq.(8) for $s_i(t)$ is valid here. Consequently, the earlier results for the minimum-fuel translational manuever are also applicable to this case.

Note that requiring the i-th and j-th spacecraft to move along their respective directions $\mathbf{w}_i(t)$ and $\mathbf{w}_j(t) = -\mathbf{w}_i(t)$ simultaneously during transposition could result in a collision. This situation can be avoided by executing a collision-avoidance manuever when the spacecraft are close to each other, or by introducing a side-stepping motion before initiating the transposition, and a recovery motion before terminating the transposition. The above-mentioned situation can also be avoided by moving more than two spacecraft simultaneously along straightline paths to their desired positions without transposition.

Type 1 Formation Reconfiguration

We assume that the formation patterns $\mathcal{P}^o(t) = \{\mathbf{r}_1^o(t), \ldots, \mathbf{r}_N^o(t)\}, t \geq 0$, before reconfiguration are shape-invariant. Let the desired formation pattern at time t be denoted by $\mathcal{P}^d(t) = \{\mathbf{d}_1(t), \ldots, \mathbf{d}_N(t)\}$. We further assume that there is a one-to-one correspondence between the elements of $\mathcal{P}^d(t)$ and the set of desired spacecraft positions $\{\mathbf{r}_1^d(t), \ldots, \mathbf{r}_N^d(t)\}$ such that $\mathbf{r}_i^d(t) = \mathbf{d}_j(t)$ for $i, j, \in \mathcal{I}$.

First, we consider the simplest case where $\mathbf{r}_i^o(t)$ and $\mathbf{r}_i^d(t) = \mathbf{d}_j(t)$ are related by a translation, i.e.

$$\mathbf{r}_i^d(t) = \mathbf{a}_i + \mathbf{r}_i^o(t), \quad i \in \mathcal{I}, \tag{22}$$

where $\mathbf{a}_i$ is a given constant vector in R_3, and $\mathcal{I}$ denotes the spacecraft index set. Evidently, $\mathcal{P}^d(t)$, and $\{\mathbf{r}_1^d(t), \ldots, \mathbf{r}_N^d(t)\}, t \geq 0$, are also shape invariant.

Assuming that $\mathbf{r}_i^d(t)$ is *visible* from $\mathbf{r}_i^o(t)$ at any time $t \geq 0$ for any $i \in \mathcal{I}$, (i.e. the line segment joining $\mathbf{r}_i^d(t)$ and $\mathbf{r}_i^o(t)$ does not pass through $\mathbf{r}_j^o(t)$ or $\mathbf{r}_j^d(t)$ for any $j \in \mathcal{I} - \{i\}$), then, given a transfer time $T_i \geq T_i^*$, the i-th spacecraft can be steered from $\mathbf{r}_i^o(0)$ to $\mathbf{r}_i^d(T_i)$ with minimum fuel expenditure by a control of the form (14). The formation reconfiguration can be achieved by moving one or a small number of spacecraft at a time. Thus, the minimum total fuel expenditure for formation reconfiguration is given by $F_T = \sum_{i=1}^{N} F(u_i^*)$. Note that given an upper bound $\bar{F}_T$ for F_T, we can always choose a set of transfer times $T_1, \ldots, T_N$ satisfying $T_i \geq T_i^*, i = 1, \ldots, N$ such that $F_T \leq \bar{F}_T$.

Next, we consider the case where $\mathbf{r}_i^d(t) = \mathbf{r}_j^o(t)$ for some $j \in \mathcal{I} - \{i\}$. Under the assumption that the initial formation patterns are shape-invariant, $(\mathbf{r}_i^d(t) - \mathbf{r}_i^o(t))$ or $(\mathbf{r}_j^o(t) - \mathbf{r}_i^o(t))$ is a constant nonzero vector for all $t \geq 0$. Again, we assume that $\mathbf{r}_j^o(t)$ is visible from $\mathbf{r}_i^o(t)$ at any time $t \geq 0$. Although it is possible to reconfigure the formation by moving all the spacecraft to their desired positions simultaneously, such an approach may result in chaos due to control system malfunctions and/or possible collisions between the spacecraft.

Therefore we propose a more prudent approach to reconfigure the formation by introducing a sequence of manuevers involving a small number of spacecraft at a time. To clarify ideas, we first consider in detail the case involving transpositions between spacecraft pairs. Thus, the optimal formation reconfiguration problem is to find a sequence of transpositions with minimum fuel expenditure.

Let σ be a permutation of the index set $\mathcal{I}$ (i.e. a one-to-one mapping of $\mathcal{I}$ onto $\mathcal{I}$ such that for $i \in \mathcal{I}, \sigma(i) = s_i \in \mathcal{I}$; and $\sigma(i) = \sigma(j)$ if and only if $i = j$) described by the usual notation

$$\sigma = \begin{pmatrix} 1 \ldots N \\ s_1 \ldots s_N \end{pmatrix}. \tag{23}$$

Let $\mathcal{G}(\mathcal{I})$ denote the set of all permutations of $\mathcal{I}$. For $\sigma, \sigma' \in \mathcal{G}(\mathcal{I})$, we define their product $\sigma\sigma'$ by $(\sigma\sigma')(i) = \sigma(\sigma'(i)), i \in \mathcal{I}$; the identity by $\mathrm{i}(i) = i$ for each $i \in \mathcal{I}$; and σ^{-1} (the inverse of σ) by $\sigma^{-1}\sigma = \sigma\sigma^{-1} = \mathrm{i}$. It can be readily verified that $\mathcal{G}(\mathcal{I})$ is a group.

Let the initial spacecraft formation be represented by a permutation of $\mathcal{I}$ denoted by

$$\sigma_o = \begin{pmatrix} 1 \ldots N \\ s_{o1} \ldots s_{oN} \end{pmatrix}. \tag{24}$$

Since the shape of the formation body is assumed to be invariant under reconfiguration, the desired formations correspond to other permutations of $\mathcal{I}$. The set of all formations corresponding to permutations of σ_o forms a group with $N!$ elements.

Now, consider a *transposition* defined by the permutation: $\tau(i') = j'$ and $\tau(j') = i'$ for $i' \neq j' \in \mathcal{I}$, and $\tau(i) = i$ whenever $i \neq i'$ and $i \neq j'$. The following property can be verified by induction:

Property A. Every permutation $\sigma \in \mathcal{G}(\mathcal{I})$ is a product of transpositions.

The above property implies that any desired reconfigured formation is attainable from any initial formation by some product of transpositions. To facilitate the computations, we use the column vector $\mathbf{s} = (s_1, \ldots, s_N)^T$ (with $s_i \in \mathcal{I}$, and $s_i = s_j$ if and only if $i = j$) to

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denote a formation or a permutation of $\mathcal{I}$. The transformation that takes $s = (s_1, \ldots, s_N)^T$ into $s' = (s_{1'}, \ldots, s_{N'})^T$ can be represented by the $N \times N$ permutation matrix $P_{(s',s)}$ in which the elements in the i-th column (for each i) are all zero except for the one in the i'-th row, which is unity. Thus, $s' = P_{(s',s)} \, s$. Evidently, any permutation group is isomorphic with the group of corresponding permutation matrices. The transposition defined earlier can be represented by a special $N \times N$ permutation matrix $T_{(i',j')}, i' \neq j' \in \mathcal{I}$, in which the (i', j')-th and (j', i')-th elements and the diagonal elements $(i \neq i', j')$ are unity, and the remaining elements are all zero.

Now, given an initial formation s_o and a desired formation s_d, there exists a permutation matrix $P_{(s_d, s_o)}$ represented by a product of transpositions $T_{(i,j)}, \ldots, T_{(i',j')}$ such that $s_d = T_{(i,j)} \circ \ldots \circ T_{(i',j')} s_o$. In general, the product of transpositions that take s_o to s_d is nonunique. Moreover, there may exist transpositions that are not admissible due to physical constraints derived from formation geometry and other considerations.

To clarify the foregoing notions, we consider a simple example.

Example 1 Consider a spacecraft triad in triangular formations with index set $\mathcal{I} = \{1, 2, 3\}$. Let the initial formation (see Fig.2) be represented by $s_o = (1, 2, 3)^T$. Then the remaining possible triangular formations generated by permuting s_o have the representations: $s_1 = (1, 3, 2)^T, s_2 = (2, 1, 3)^T, s_3 = (2, 3, 1)^T, s_4 = (3, 2, 1)^T$, and $s_5 = (3, 1, 2)^T$. Evidently, s_1 can be generated from s_o by applying the transposition represented by the matrix:

$$T_{(2,3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \tag{25}$$

Formation s_5 can be generated from s_o by applying the permutation matrix:

$$P_{(s_5, s_o)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{26}$$

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which has three distinct decompositions in the form of products of two transpositions given by:

$$\mathbf{P}_{(\mathbf{s}_5, \mathbf{s}_o)} = \mathbf{T}_{(3,1)} \mathbf{T}_{(1,2)} = \mathbf{T}_{(2,3)} \mathbf{T}_{(3,1)} = \mathbf{T}_{(1,2)} \mathbf{T}_{(2,3)}, \tag{27}$$

where

$$\mathbf{T}_{(3,1)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad \mathbf{T}_{(1,2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{28}$$

Now, suppose that the initial formation is a *line* formation represented by $\mathbf{s}_o = (1, 2, 3)^T$ as before. The reconfigured formations are line formations generated by permuting $\mathbf{s}_o$ as shown in Fig.3. Clearly, these line formations have the same representations as in the triangular formation case. Suppose that a transposition between a spacecraft pair is *admissible* only if they are mutually visible (i.e. there are no other spacecraft lying in the line segment joining the spacecraft pair). Then, the transposition $\mathbf{T}_{(3,1)}$ is not admissible. Consequently, $\mathbf{P}_{(\mathbf{s}_5, \mathbf{s}_o)}$ has a unique admissible decomposition $\mathbf{T}_{(1,2)} \mathbf{T}_{(2,3)}$.

To solve the minimum-fuel formation reconfiguration problem, we propose the following basic steps:

Step 1. Construct the $N \times N$ minimal-fuel expenditure matrices $\mathbf{F}^{(n)}, n = 1, \ldots, N$, whose (i, j)-th element $F_{ij}^{(n)}$ corresponds to the minimal fuel expenditure associated with moving the n-th spacecraft from the j-th to the i-th location in the formation. Obviously, $\mathbf{F}^{(n)}$ is symmetric with zero diagonal elements. The numerical values for $F_{ij}^{(n)}$ can be computed using (16). For a transposition manuever involving moving the n-th spacecraft from the j-th to the i-th position in the formation, and the m-th spacecraft from the i-th to the j-th position in the formation, the total fuel expenditure is $F_{ij}^{(n)} + F_{ji}^{(m)}$.

Step 2. Given the initial and desired formations represented by $\mathbf{s}_o$ and $\mathbf{s}_d$ respectively, obtain the permutation matrix $\mathbf{P}_{(\mathbf{s}_d, \mathbf{s}_o)}$ and find all its possible decompositions in the form of products of admissible transpositions.

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Step 3. For each decomposition obtained in Step 2, compute the corresponding total fuel expenditure by summing the fuel expenditures for all the transpositions in the decomposition.

Step 4. Determine the minimum-fuel product of transpositions for formation reconfiguration by minimizing the total fuel expenditure over all admissible products of transpositions.

The first three steps are straightforward. The fourth step is akin to the well-known "Travelling Salesman Problem". When the number of spacecraft is large, say $N \geq 50$, the solution to this problem becomes computationally intensive. However, the presence of constraints helps to reduce the total number of product of admissible transpositions thereby reducing the complexity. Moreover, efficient methods for solving this problem are available [12]. Note that all the foregoing steps can be performed *off-line*, and the resulting optimal sequences of spacecraft transpositions can be stored in computers on-board the spacecraft.

As mentioned earlier, collision between a pair of spacecraft during transposition may occur without collision-avoidance manuevers. Therefore, it is of interest to consider moving small sets of spacecraft in sequence, each set containing more than two spacecraft, to achieve formation reconfiguration. To facilitate the subsequent development, we introduce a few definitions.

Definition 1. Let σ and σ' be two permutations of $\mathcal{I}$. Then σ and σ' are *disjoint* if every integer in $\mathcal{I}$ moved by σ is fixed by σ', and every integer moved by σ' is fixed by σ.

Definition 2. A permutation σ of $\mathcal{I}$ is a *cycle of length m* or *m-cycle*, if it is a permutation of a subset $\{s'_1, \ldots, s'_m\}$ of $\mathcal{I}$ that replaces s'_1 by s'_2, s'_2 by $s'_3, \ldots, s'_{m-1}$ by s'_m, and s'_m by s'_1.

Evidently, two disjoint permutations σ and σ' commute, i.e. $\sigma\sigma' = \sigma'\sigma$. Also, a transposition is a cycle of length 2. We know that given any initial and desired formations

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corresponding to permutations of $\mathcal{I}$ represented by $\mathbf{s}_o$ and $\mathbf{s}_d$ respectively, there exists a permutation matrix $\mathbf{P}_{(\mathbf{s}_d, \mathbf{s}_o)}$ which is decomposable into a product of transpositions. Now, the main question is whether it is possible to decompose $\mathbf{P}_{(\mathbf{s}_d, \mathbf{s}_o)}$ into a product of *permutation cycles of length* $m > 2$. To answer the foregoing question, we make use of the following basic property of permutation groups [13],[14]:

Property B. Let σ be a permutation of $\mathcal{I}$. Then σ can be expressed as a product of disjoint cycles. This cycle decomposition is unique up to re-arrangement of the cycles involved.

From Definitions 1 and 2, disjoint cycles imply that no two cycles move a common point. Note that the cycles in the decomposition may have lengths ≥ 2. Property B suggests that we should first seek the decompositions of the given $\mathbf{P}_{(\mathbf{s}_d, \mathbf{s}_o)}$ in terms of disjoint cycles which can be readily determined (for a simple algorithm, see page 30 of [14]). If all the disjoint cycles have lengths > 2, then we can reconfigure the formation by a composition of permutations of small spacecraft groups containing more than two spacecraft. For an admissible product of permutation cycles, the corresponding minimum fuel expenditure can be determined from (16). In the actual formation reconfiguration process, all the spacecraft associated with any cycle must move simultaneously. But the disjoint cycles can be initiated at different times. The foregoing approach can be illustrated by the following example.

Example 2 Consider four spacecraft in diamond-shaped formation patterns. Here, the index set $\mathcal{I}$ is $\{1, 2, 3, 4\}$. Let the initial formation (see Fig.4) be represented by $\mathbf{s}_o = (1, 2, 3, 4)^T$. The index set $\mathcal{I}$ has 4! permutations including the identity permutation. These permutations can be classified into the following categories:

6 cycles of length 4:

$$\mathcal{C}_4 = \{(4, 1, 2, 3)^T, (2, 3, 4, 1)^T, (2, 4, 1, 3)^T, (3, 1, 4, 2)^T, (3, 4, 2, 1)^T, (4, 3, 1, 2)^T\};$$

8 cycles of length 3:

$$C_3 = \{(4,1,3,2)^T, (2,4,3,1)^T, (3,1,2,4)^T, (2,3,1,4)^T, (3,2,4,1)^T, (4,2,1,3)^T,$$
$$(1,3,4,2)^T, (1,4,2,3)^T\};$$

6 cycles of length 2:

$$C_2 = \{(2,1,3,4)^T, (4,2,3,1)^T, (1,2,4,3)^T, (1,3,2,4)^T, (3,2,1,4)^T, (1,4,3,2)^T\};$$

and 3 non-cycles:

$$\mathcal{N} = \{(2,1,4,3)^T, (4,3,2,1)^T, (3,4,1,2)^T\}.$$

The formations corresponding to these permutations are illustrated in Fig.4. The spacecraft movements are indicated by arrows. Now, if the desired spacecraft formation s_d belongs to C_4 or C_3, then formation reconfiguration can be achieved by a single cyclic permutation without transposition. The total fuel expenditure can be computed by summing the fuel expenditures associated with the movements of spacecraft in the permutation cycle using (16). If s_d belongs to C_2 or $\mathcal{N}$, then formation reconfiguration can be obtained by transpositions or product of cycles of lengths > 2. For example, the desired formation $s_d = (3,4,1,2)^T \in \mathcal{N}$ can be attained from the initial formation $s_o = (1,2,3,4)^T$ by two disjoint transpositions: $s_o \rightarrow (3,2,1,4)^T \rightarrow s_d$, or by two successive nondisjoint cycles of length 3: $s_o \rightarrow (3,2,4,1)^T \rightarrow s_d$. We note from Fig.4 that for the first four cycles of length 4, collision between spacecraft occurs when the paths connecting the 1,2 and 3,4 positions in the formation cross each other at the same time. However, collision may be avoided by making the transition times associated with the paths connecting positions 1,2 and 3,4 different from each other.

Type 2 Formation Reconfiguration

In this type of formation reconfiguration, a specified position in the desired formation may be occupied by *any* spacecraft of a particular type. As before, the initial and desired formation patterns at time t are specified by the point sets $\mathcal{P}^o(t) = \{r_1^o(t), \ldots, r_N^o(t)\}$

and $\mathcal{P}^d(t) = \{\mathbf{d}_1(t), \ldots, \mathbf{d}_N(t)\}$ respectively. We assume that each element $\mathbf{d}_j(t) \in \mathcal{P}^d(t)$ corresponds to a unique $\mathbf{r}_i^d(t)$ for some $i \in \mathcal{I} = \{1, \ldots, N\}$.

First, we consider the simplest case where all N spacecraft are identical. Moreover, $\mathbf{r}_i^o(t)$ and $\mathbf{r}_i^d(t) = \mathbf{d}_j(t)$ are related by a translation given by (22). In this case, there are $N!$ permutations for formation reconfiguration. However some permutations may not be admissible due to the fact that $\mathbf{r}_i^d(t)$ is not visible from $\mathbf{r}_i^o(t)$ for some $i \in \mathcal{I}$. The solution to the minimum-fuel reconfiguration problem can be solved by first computing the total minimum-fuel expenditure associated with each admissible permutation, and then determine the permutation with the least total minimum-fuel expenditure.

Next, we consider the general case where $\mathbf{r}_i^d(t)$ and $\mathbf{r}_i^o(t)$ have the form:

$$\mathbf{r}_i^d(t) = \mathbf{r}_i^d(0) + \mathbf{q}(t), \qquad \mathbf{r}_i^o(t) = \mathbf{r}_i^o(0) + \mathbf{q}(t), \qquad i = 1, \ldots, N, \tag{29}$$

where $\mathbf{q}(t)$ is a specified formation drift vector. Thus, the initial and desired formation patterns are shape-invariant over some time interval I_T. Let $\mathcal{P}^o(t)$ (resp. $\mathcal{P}^d(t)$) be partitioned into P disjoint subsets of identical spacecraft represented by index set S_j^o (Resp. S_j^d) $\subset \mathcal{I}$ with $\cup_{j=1}^P S_j^d = \cup_{j=1}^P S_{j=1}^o = \mathcal{I}$. We assume that S_j^o and S_j^d have the same number of elements n_j. Since any element of S_j^d may be replaced by any element of S_j, therefore we have a total of $N!/\Pi_{j=1}^P(n_j!)$ distinct formations. Again, we use the column vector $\mathbf{s}_j = (s_{j1}, \ldots, s_{jn_j}^T)$ with $s_{ji} \in S_j$, and $\boldsymbol{s}_{ji} = \boldsymbol{s}_{ji'}$ if and only if $i = i'$, to denote a permutation of S_j. Thus, the initial and desired formations can be represented by $\mathbf{s}_o = (\mathbf{s}_{o1}^T, \ldots, \mathbf{s}_{oP}^T)^T$ and $\mathbf{s}_d = (\mathbf{s}_{d1}^T, \ldots, \mathbf{s}_{dP}^T)^T$ respectively. Given $\mathbf{s}_o$ and $\mathbf{s}_d$, the permutation matrix $\mathbf{P}_{(\mathbf{s}_d, \mathbf{s}_o)}$ relating $\mathbf{s}_o$ and $\mathbf{s}_d$ may not be unique, since there may exist many ways for attaining $\mathbf{s}_{di}$ by the components of $\mathbf{s}_{oi}$. For a given $\mathbf{P}_{(\mathbf{s}_d, \mathbf{s}_o)}$, we may seek its decomposition as a product of disjoint cycles as proposed earlier for Type 1 formation reconfiguration, and determine the minimum-fuel decomposition corresponding to $\mathbf{P}_{(\mathbf{s}_d, \mathbf{s}_o)}$. Then the solution to the minimum-fuel formation reconfiguration

problem can be found by considering the minimum-fuel admissible decompositions for all possible permutations associated with $\mathbf{P}_{(\mathbf{s}_d, \mathbf{s}_o)}$.

To obtain a practical solution to the foregoing problem, we propose to simplify the problem by first identifying those spacecraft in the initial formation that match both the positions and spacecraft types in the desired formation. These spacecraft will remain *fixed* relative to the formation during reconfiguration. Then, we seek a solution to the minimum-fuel reconfiguration problem for the *remaining* spacecraft using the method described earlier. This simplified approach may lead to a sub-optimal solution when the fuel expenditures for moving the fixed spacecraft are small relative to those for moving the remaining spacecraft. Now, we illustrate the application of the proposed method with the aid of an example.

Example 3. Consider eight spacecraft whose initial and desired formations correspond to equally spaced points on a circle (see Fig.5). There are two sets of spacecraft (marked by small circles and black dots). Each set contains four identical spacecraft. Let the initial formation be specified by the pattern shown in Fig.5, and represented by $\mathbf{s}_o = (1, 2, \ldots, 8)^T$. Since no distinction is made between identical spacecraft, hence there are $8!/(4!)^2 = 70$ possible formations generated by permutations. Let the desired formation be specified by the pattern shown in Fig.5. We observe that the spacecraft at the 2,3,7 and 8-th positions in the initial formation match the types of spacecraft in the corresponding positions in the desired formation. Following the proposed approach, we first consider moving only spacecraft 1,4,5 and 6 to attain the desired formation. In particular, we seek permutation cycles of $\{1, 4, 5, 6\}$ with lengths > 2 that match the corresponding points in the desired formation. Figure 5 shows two permutation cycles of length 4 which lead to the desired formation, namely $(6, 2, 3, 1, 4, 5, 7, 8)^T$ or $(4, 2, 3, 5, 6, 1, 7, 8)^T$. It is evident that there are no permutation cycles of length 3 that lead to the desired formation, if we only allow to move spacecraft 1,4,5 and 6. But if we allow to move spacecraft 1,2,4,5,6 and 8, while keeping

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spacecraft 3 and 7 fixed relative to the formation, then the desired formation can be attained by any one of the following products of two disjoint cycles of length 3 (see Fig.6):

$$\{(1,2,3,4,8,5,7,6)^T, (2,4,3,1,8,5,7,6)^T\},$$

$$\{(8,2,3,4,5,1,7,6)^T, (8,4,3,5,2,1,7,6)^T\},$$

$$\{(2,6,3,4,5,1,7,8)^T, (2,6,3,5,8,1,7,4)^T\},$$

$$\{(8,2,3,1,5,6,7,4)^T, (8,6,3,1,2,5,7,4)^T\}.$$

Thus, if we choose to fix spacecraft 3 and 7 relative to the formation, and move only three spacecraft at a time, the minimum-fuel formation reconfiguration can be determined by computing the fuel expenditures for all the products of disjoint cycles of length 3 (assuming that all these products are admissible), and selecting the one with lowest fuel expenditure.

Simulation Study

The main objective of the simulation study is to determine the dynamics of the spacecraft with the proposed methods for minimum fuel formation reconfiguration. Here, each spacecraft is equipped with a simple collision avoidance control. Let $\mathcal{B}_o = \{e_1, e_2, e_3\}$ be an orthonormal basis for the three-dimensional Euclidean space R^3. The representation of any position vector $r \in R^3$ with respect to $\mathcal{B}$ is denoted by the column vector $(x, y, z)^T$.

First, we consider four nonidentical spacecraft with diamond-shaped formation patterns as in Example 2. At time $t = 0$, all spacecraft lie in the plane $\{(x, y, z) \in R^3 : z = 0\}$. The initial formation patterns are specified by $\mathcal{P}^o(t) = \{r_1^o(t), \ldots, r_4^o(t)\}$, where

$$[r_1^o(t)]_o = (50, 10, 0)^T + q(t), \qquad [r_2^o(t)]_o = (50, 50, 0)^T + q(t),$$

$$[r_3^o(t)]_o = (10, 30, 0)^T + q(t), \qquad [r_4^o(t)]_o = (90, 30, 0)^T + q(t),$$

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where the coordinates are in meters. We assume that all spacecraft drift upward along the z-axis with a constant speed of 10^3 m/sec in free space in the absence of gravity and external forces. Thus, $\mathbf{q}(t) = (0, 0, 10^3 t)^T$. The masses and the magnitude bounds for the controls of spacecraft 1-4 are given by $10, 20, 30, 40$ kg; and $0.1, 0.2, 0.3, 0.4$ N. respectively. Let the desired formation pattern at time t be specified by $\mathcal{P}^d(t) = \{\mathbf{d}_1(t), \ldots, \mathbf{d}_4(t)\}$, where

$$[\mathbf{d}_1(t)]_o = [\mathbf{r}_3^o(t)]_o, \quad [\mathbf{d}_2(t)]_o = [\mathbf{r}_4^o(t)]_o, \quad [\mathbf{d}_3(t)]_o = [\mathbf{r}_1^o(t)]_o, \quad [\mathbf{d}_4(t)]_o = [\mathbf{r}_2^o(t)]_o. \quad (30)$$

Assuming that $\mathbf{r}_i^d(t) = \mathbf{d}_i(t)$ for $i = 1, 2, 3, 4$, the desired and initial formations can be represented by $\mathbf{s}_d = (3, 4, 1, 2)^T$ and $\mathbf{s}_o = (1, 2, 3, 4)^T$ respectively, and they are related by a permutation represented by the matrix

$$\mathbf{P}_{(\mathbf{s}_d, \mathbf{s}_o)} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \quad (31)$$

The formation reconfiguration can be achieved by two disjoint transpositions between spacecraft 1 and 3, and between spacecraft 2 and 4. To avoid collisions, we use the following simple scheme similar to that described in [14]. Let each spacecraft be enclosed by a ball $S_i(t)$ centered ar $\mathbf{r}_i(t)$ with specified radius ϵ_i (i.e. $S_i(t) = \{\mathbf{r} \in R^3 : \|\mathbf{r} - \mathbf{r}_i(t)\| \leq \epsilon_i\}$). The i-th and j-th spacecraft are said to be within *unsafe range* at time t, if $S_i(t) \cap S_j(t)$ is nonempty. In this situation, we set the controls for the i-th and j-th spacecraft to

$$f_{ci}(t) = \bar{f}_{ci} u_i^*(t) \mathbf{w}_i(t) + \alpha_i \mathbf{h}_i(t), \qquad f_{cj}(t) = \bar{f}_{cj} u_j^*(t) \mathbf{w}_j(t) + \alpha_j \mathbf{h}_j(t), \quad k = 1, 2, 3, \quad (32)$$

where $\mathbf{w}_i(t)$ is defined in (3); $\mathbf{h}_i(t)$ is a unit vector orthogonal to $\mathbf{r}_j(t) - \mathbf{r}_i(t)$; $\mathbf{h}_j(t) = -\mathbf{h}_i(t)$; α_i and α_j are suitable positive constants; $\bar{f}_{ci}$ and $\bar{f}_{cj}$ are given positive constants corresponding to the maximum thrust magnitudes for the i-th and j-th spacecraft respectively; and u_i^* is given by (17) with parameter β set at 1.1. Figure 7 shows the spacecraft trajectories in R^3 during formation reconfiguration without collision-avoidance controls. The time-domain

records for the corresponding minimum-fuel controls are shown in Fig.8. The normalized fuel expenditures for each spacecraft and the total normalized fuel expenditure are shown in Figs.9a and 9b respectively. Here the transposition between spacecraft 1 and 3 (resp. 2 and 4) takes place for $100 \leq t \leq 242.45$ sec. (resp. $350 \leq t \leq 492.45$ sec.) Next, the collision-avoidance controls given by (32) with $\epsilon = 20$ m. and $\mathbf{h}_i(t) = (0, 0, 1)^T$ are activated. The corresponding spacecraft trajectories in R^3 are essentially identical to those shown in Fig.7 except for the period when the collision-avoidance controls are active. The variation of the distance between spacecraft pair $\{1, 3\}$ (or $\{2, 4\}$) with time during this period is shown in Fig.10. Figure 11 shows the corresponding normalized total fuel expenditure. It can be seen that the total fuel expenditure is increased approximately 50 percent over that for the case without collision-avoidance controls.

Now we consider eight spacecraft composed of two sets of identical spacecraft, $S_1 = \{1, 3, 5, 7\}$ and $S_2 = \{2, 4, 6, 8\}$ with 10 kg. and 20 kg. masses; and with control magnitude bounds 0.1, 0.2 N. respectively. Their initial and desired formations correspond to equally spaced points on a circle as in Example 3. At time $t = 0$, all spacecraft lie in the plane $\{(x, y, z) \in R^3 : z = 0\}$. The initial formation patterns are specified by $\mathcal{P}^o(t) = \{\mathbf{r}_1^o(t), \dots, \mathbf{r}_8^o(t)\}$, where

$$[\mathbf{r}_i^o(t)]_o = (60 + 50 \cos((3-i)\pi/4), 60 - 50 \sin((3-i)\pi/4), 0)^T + \mathbf{q}(t), \qquad i = 1, \dots, 8, \quad (33)$$

where $\mathbf{q}(t) = (0, 0, 10^3 t)^T$ implying that all spacecraft drift upward along the z-axis with a constant speed of 10^3 m/sec.

First, we consider formation reconfiguration by cycling two disjoint sets of three spacecraft. From Fig.6, it is evident that the fuel expenditures associated with the first two products of disjoint cycles of length 3 are identical. This is also true for the last two products of disjoint cycles of length 3. Figure 12 shows the spacecraft trajectories in R^3 during

reconfiguration by cycling spacecraft 1,2,4 and 5,6,8. The normalized fuel expenditures for each spacecraft and the normalized total fuel expenditure are shown in Figs.13 and 14 respectively. Similar results are obtained for the case where reconfiguration is achieved by cycling spacecraft 1,2,6 and 4,5,8. Their corresponding spacecraft trajectories and normalized total fuel expenditures are shown in Figs.15 and 16 respectively. Finally, we consider formation reconfiguration by cycling four spacecraft to achieve the desired formation as depicted in Fig.5. Clearly, both cycles of length 4 as shown in Fig.5 have the same fuel expenditure. The spacecraft trajectories and the normalized total fuel expenditures are shown in Figs.17 and 18 respectively. Comparing the total fuel expenditures for all foregoing cases, we conclude that the minimum total fuel expenditure for formation reconfiguration is attained by cycling four spacecraft $\{1, 4, 5, 6\}$.

Concluding Remarks

The main idea in the proposed method for solving the minimum-fuel formation reconfiguration problem is to break up the reconfiguration process into a sequence of basic minimum-fuel maneuvers involving a small number of spacecraft. Thus, it represents a simple prudent approach to the formation reconfiguration process in which catastrophic failures could be avoided. The method requires determining first the basic minimum-fuel maneuvers for a single spacecraft. Then, the minimum-fuel formation reconfiguration problem is reduced to a combinatorial optimization problem. In this paper, we have considered only the case where no significant gravitational and other disturbance forces are present so that only straightline spacecraft trajectories need to tbe considered. When the gravitational forces are significant as in the case where the spacecraft move along low planetary orbits, there may exist regimes where the gravitational forces may be utilized to reduce fuel expenditure during formation reconfiguration. In this case, the minimum-fuel trajectories are generally *space curves*. Nevertheless, the proposed approach remains applicable once the minimum-fuel maneuvers for a

single spacecraft have been determined. In fact, the proposed approach is also applicable for optimization problems involving general cost functionals other than total fuel expenditure [16],[17].

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Figure Captions

Fig.1 Minimum-fuel trajectory.

Fig.2 Triangular formation.

Fig.3 Line formations for spacecraft triad.

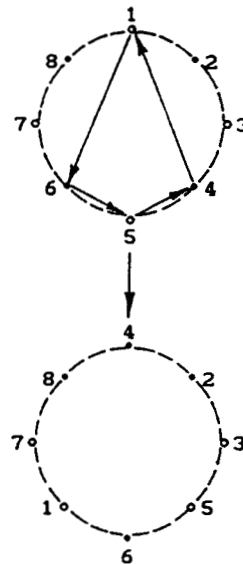Fig.4 Decomposition of formation for four spacecraft into cycles.

Fig.5 Formation consisiting of eight spacecraft equally spaced on a circle with fixed spacecraft 2,3,7 and 8.

Fig.6 Products of disjoint ccycles of length 3 corresponding to eight spacecraft with fixed spacecraft 3 and 7 for attaining s_d.
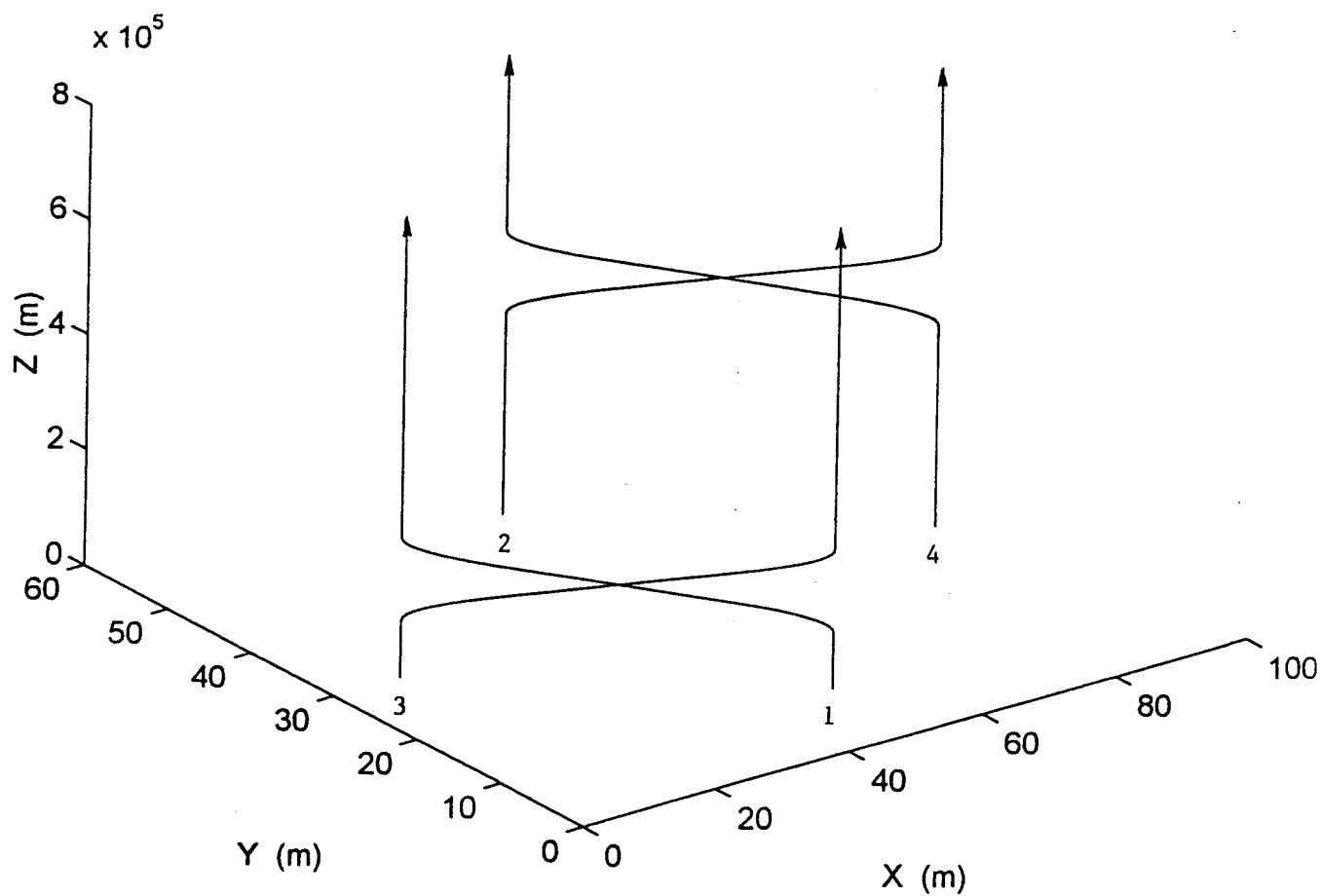
Fig.7 Trajectories of four spacecraft in R^3 during reconfiguration involving transposition between spacecraft 1,3 and 2,4 without collision-avoidance controls.

Fig.8 Time-domain behavior of minimum-fuel controls for spacecraft 1-4.

Fig.9a Normalized fuel expenditures for spacecraft 1-4 during reconfiguration.

Fig.9b Total normalized fuel expenditure during reconfiguration.

Fig.10 Distance between spacecraft pair $\{1,3\}$ (or $\{2,4\}$) during collision-avoidance maneuver.

Fig.11 Total normalized fuel expenditure during reconfiguration.

Fig.12 Trajectories of two sets of four identical spacecraft in R^3 during reconfiguration by cycling spacecraft 1,2,4 and 5,6,8.

Fig.13a Normalized fuel expenditures for spacecraft 1,2 and 4 during reconfiguration.

Fig.13b Normalized fuel expenditures for spacecraft 5,6 and 8 during reconfiguration.

Fig.14 Total normalized fuel expenditure during reconfiguration by cycling spacecraft 1,2,4

and 5,6,8.

Fig.15 Trajectories of two sets of four identical spacecraft in R^3 during reconfiguration by cycling spacecraft 1,2,6 and 4,5,8.

Fig.16 Total normalized fuel expenditure during reconfiguration by cycling spacecraft 1,2,6 and 4,5,8.

Fig.17 Trajectories of two sets of four identical spacecraft in R^3 during reconfiguration by cycling spacecraft 1,4,5 and 6.

Fig.18 Total normalized fuel expenditure during reconfiguration by cycling spacecraft 1,4,5 and 6.

Fig. 1 Minimum-fuel trajectory.

$$s_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad s_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad s_4 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad s_5 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$
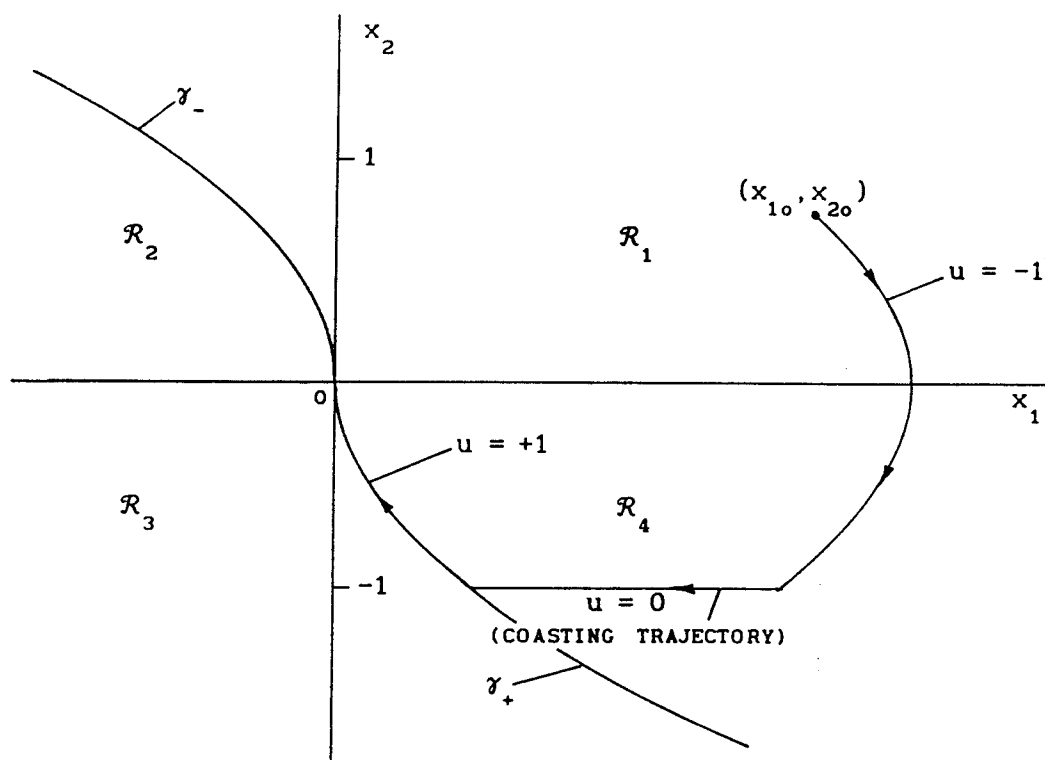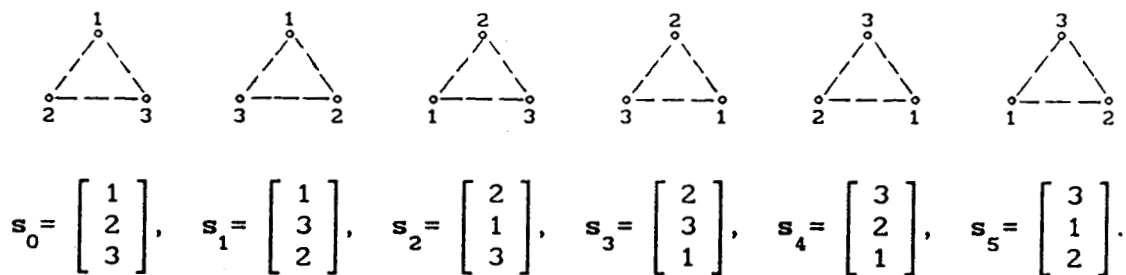
(a) All possible triangular formations.
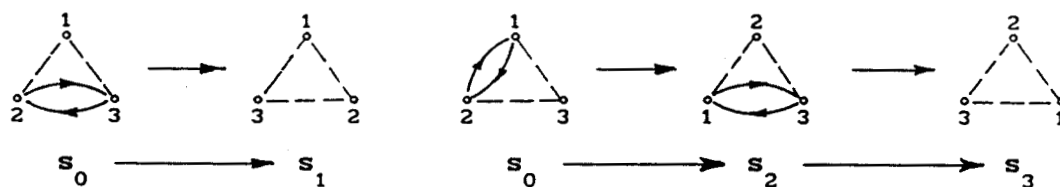
(b) Formations attained by transpositions.

Fig.2 Triangular formations.

$$s_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad s_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad s_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad s_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \quad s_4 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad s_5 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

(a) All possible line formations.

$$s_0 \longrightarrow s_1 \longrightarrow s_5$$

ADMISSIBLE TRANSPOSITION

$$s_0 \longrightarrow s_2 \longrightarrow s_5 \qquad s_0 \longrightarrow s_4 \longrightarrow s_5$$

INADMISSIBLE
TRANSPOSITIONS
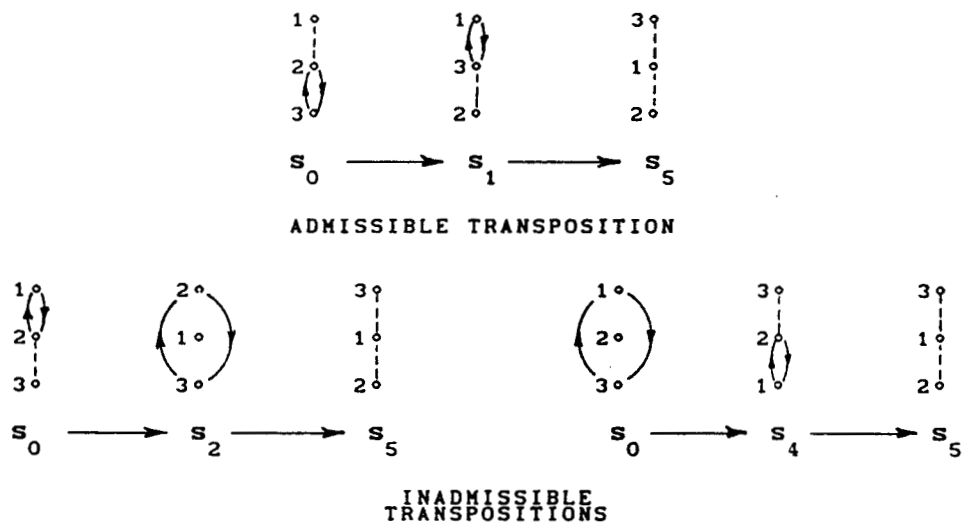
(b) Admissible and inadmissible transpositions.

Fig. 3 Line formations for spacecraft triad.

· INITIAL FORMATION:

$$s_0 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix};$$
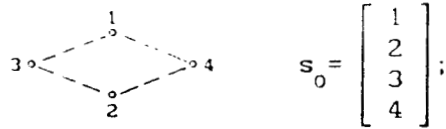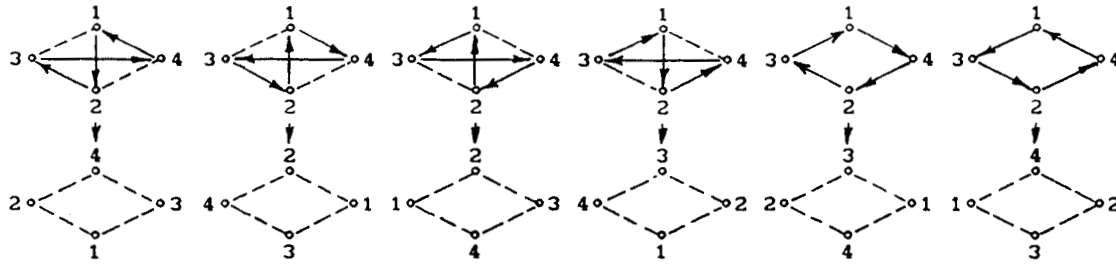
· CYCLES OF LENGTH 4:

·. CYCLES OF LENGTH 3:
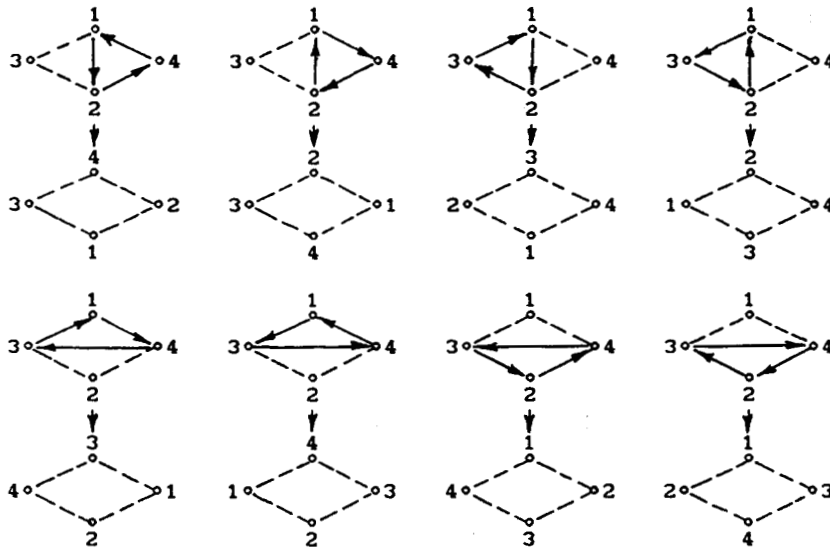
·. CYCLES OF LENGTH 2 (TRANSPOSITION):
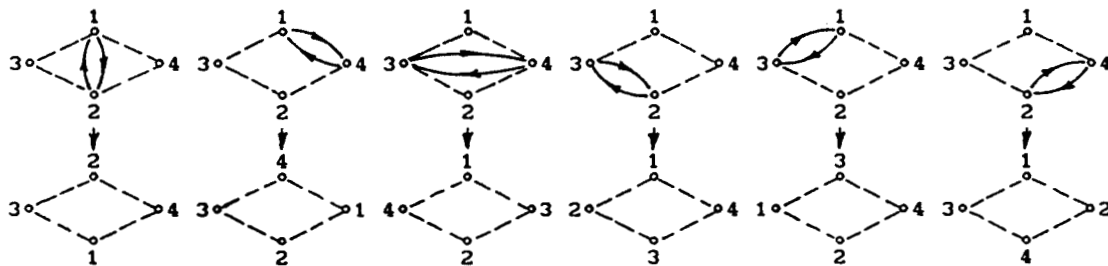
·. NON-CYCLES:
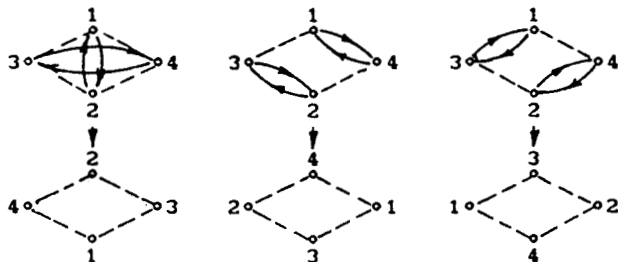
Fig. 4 Decomposition of formations for four spacecraft into cycles.

- Initial Formation s_o :
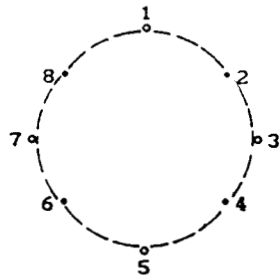
$$s_o = (1, 2, \ldots, 8)^T$$

• Desired Formation s_d :
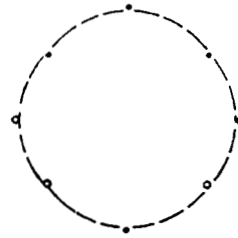
• Cycles of Length 4 for Attaining s_d :
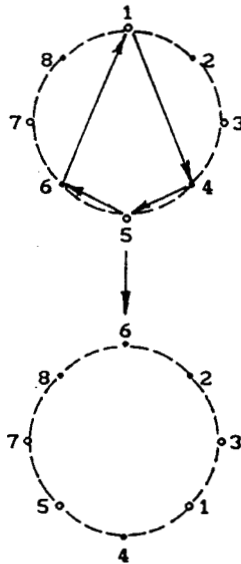
Fig. 5 Formation consisting of eight spacecraft equally spaced on a circle with fixed spacecraft 2, 3, 7 and 8.
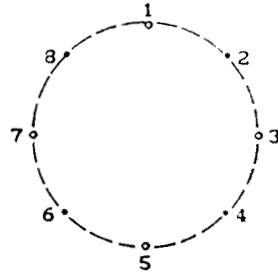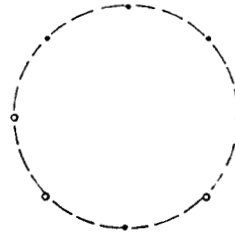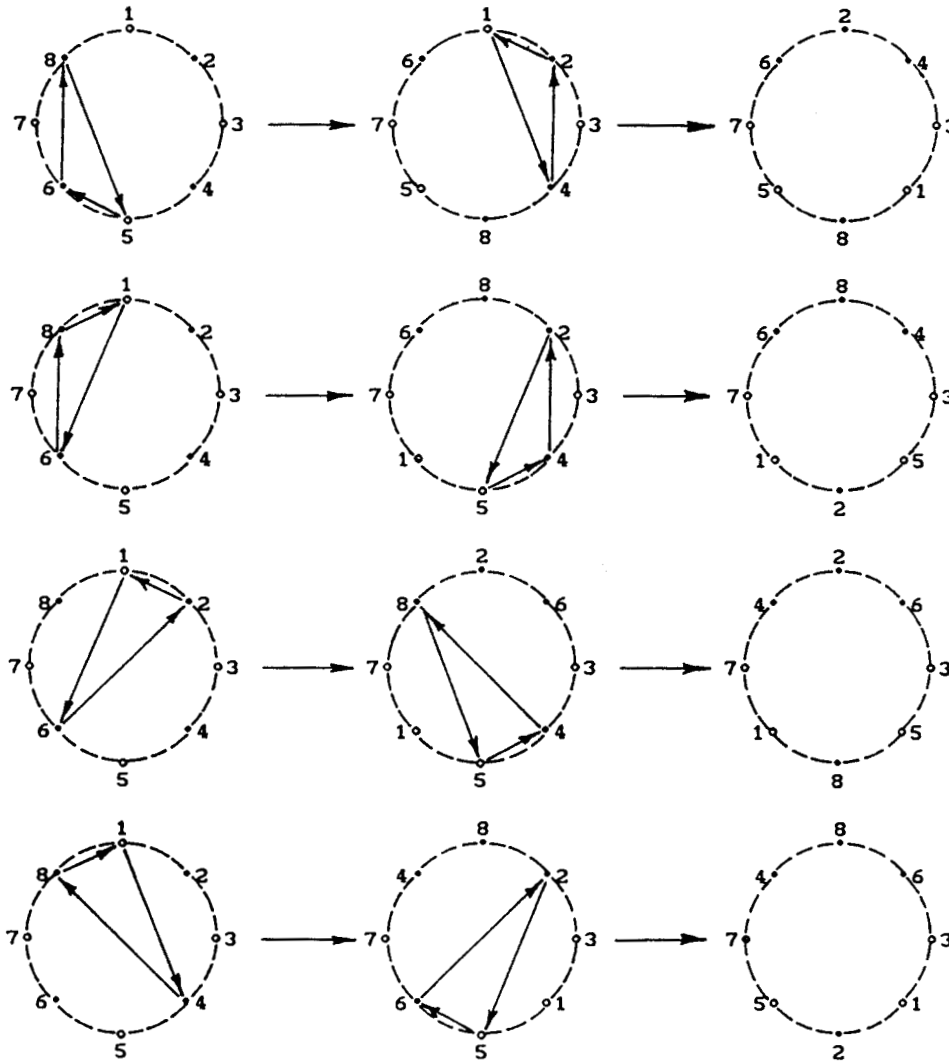
Fig. 6 Products of disjoint cycles of length 3 corresponding to 8 spacecraft with fixed spacecraft 3 and 7 for attaining s_d.

SPACECRAFT MOTION IN 3D-SPACE

Fig. 7 Trajectories of four spacecraft in the world space during reconfiguration involving transposition between spacecraft 1,3 and 2,4, without collision-avoidance controls.

Fig. 8 Time-domain behavior of minimum-fuel controls for spacescraft 1-4.

Fig. 9a Normalized fuel expenditures for spacecraft 1-4 during
reconfiguration.
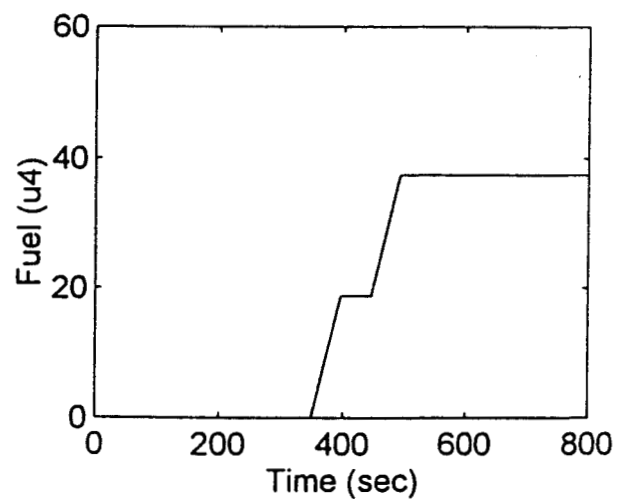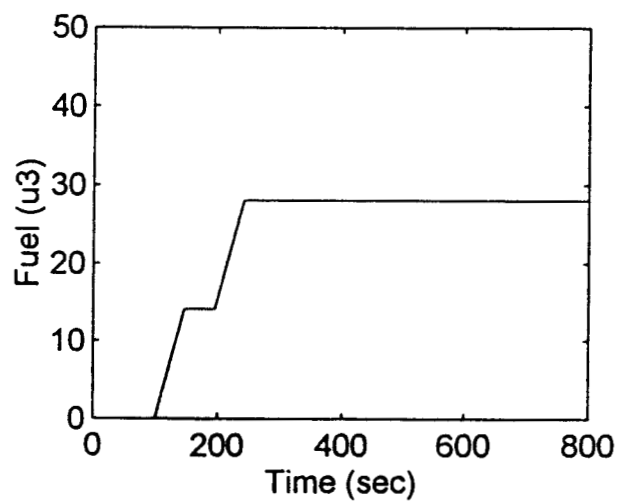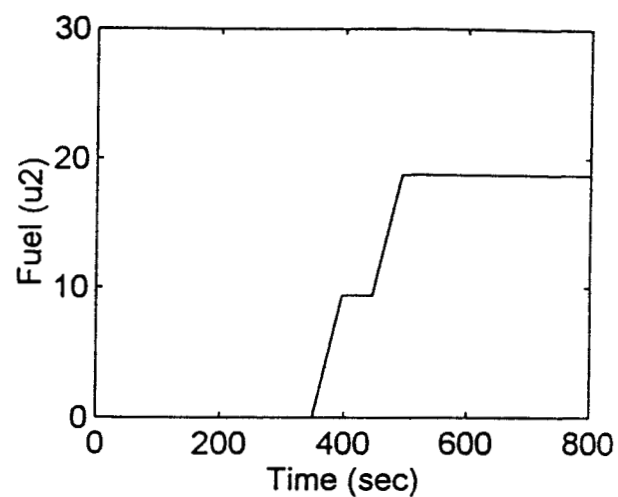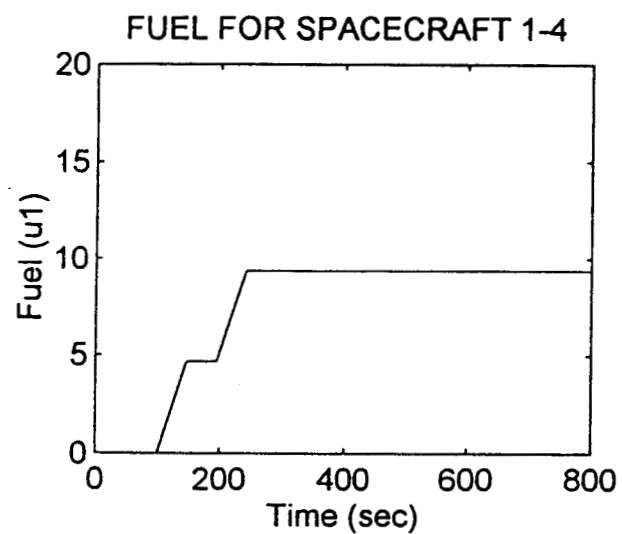
TOTAL FUEL EXPENDITURE
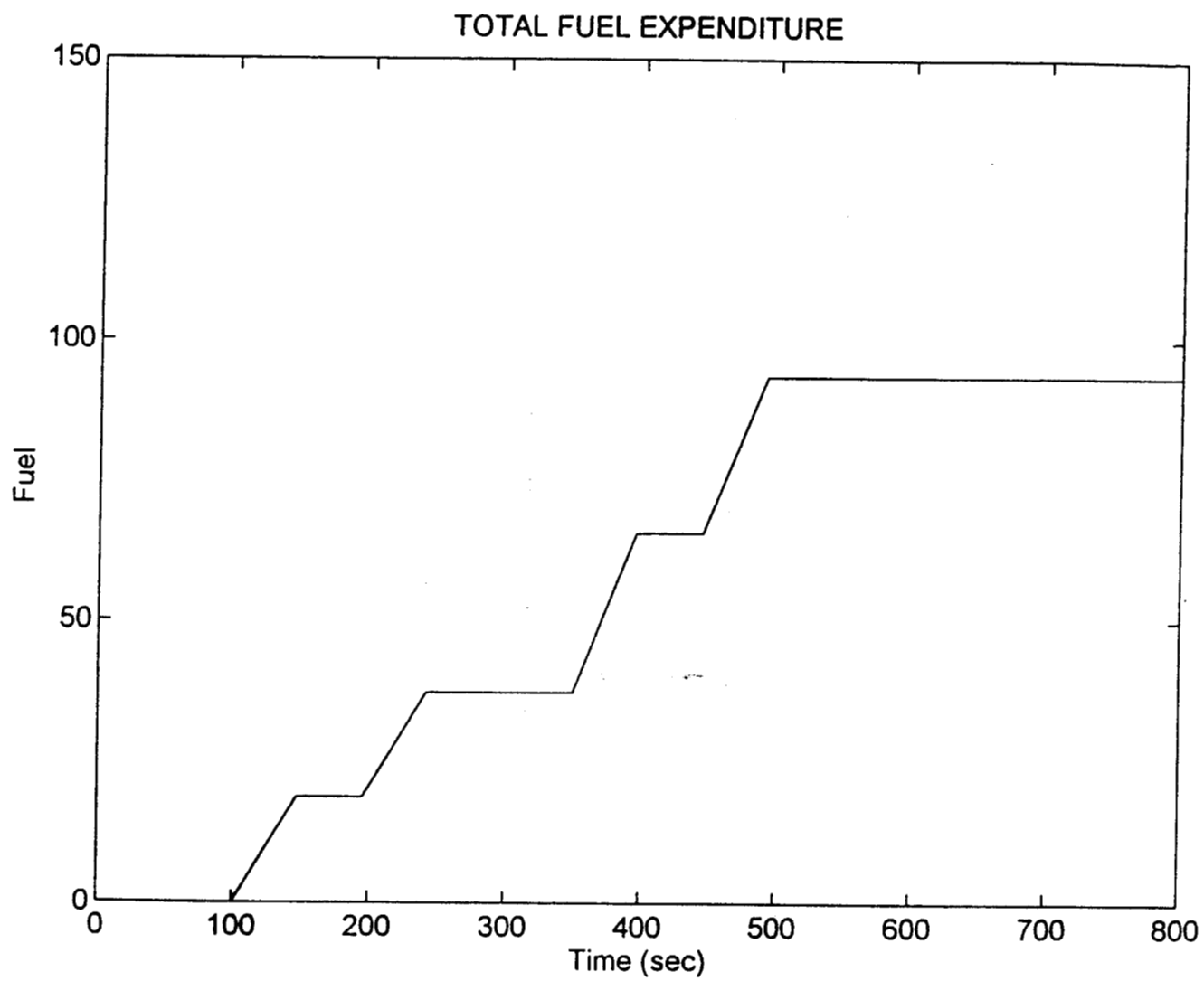
Fig. 9b Total normalized fuel expenditure during reconfiguration.
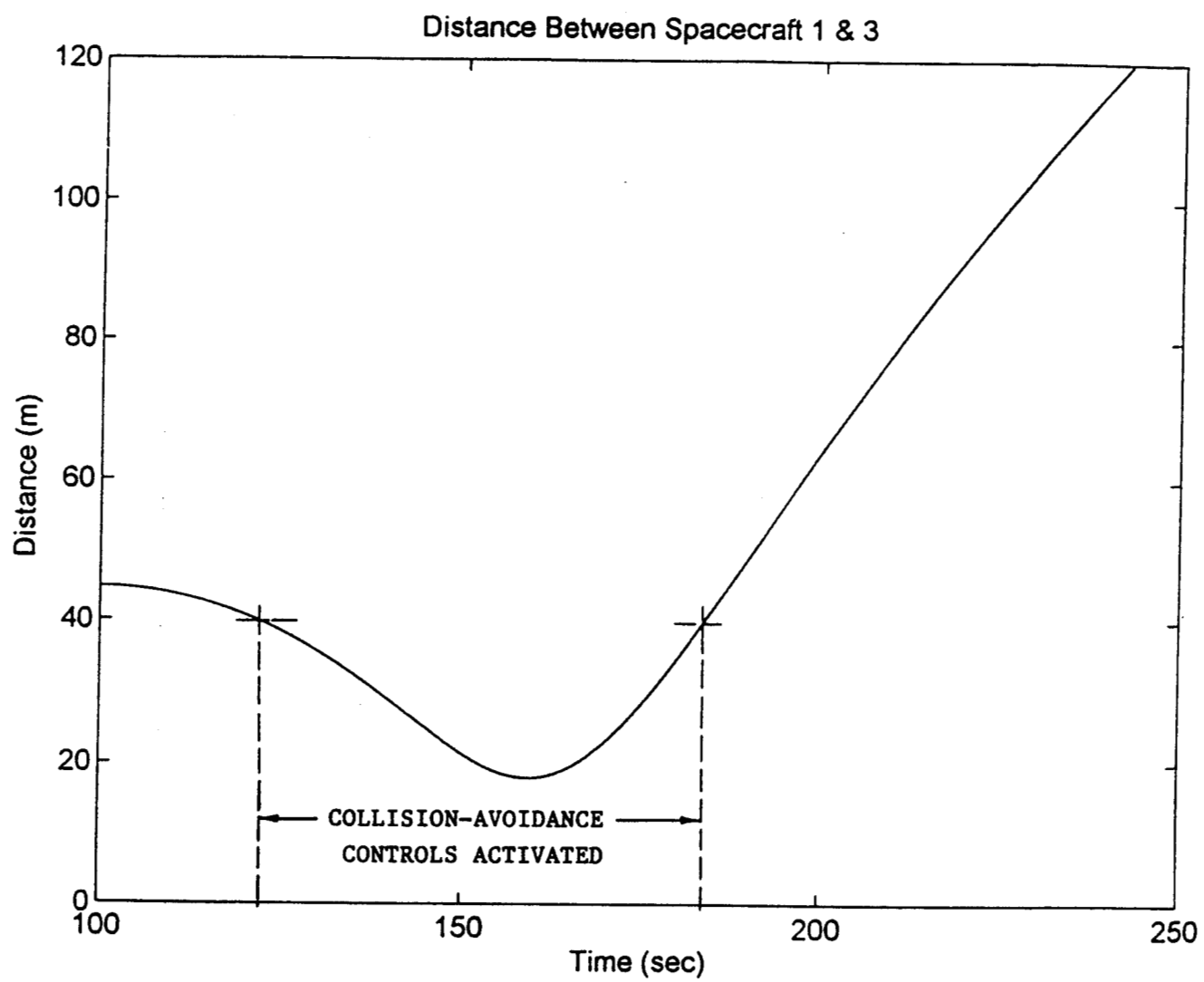
Fig.10 Distance between spacecraft pair {1, 3} (or {2, 4}) during collision-avoidance maneuver.
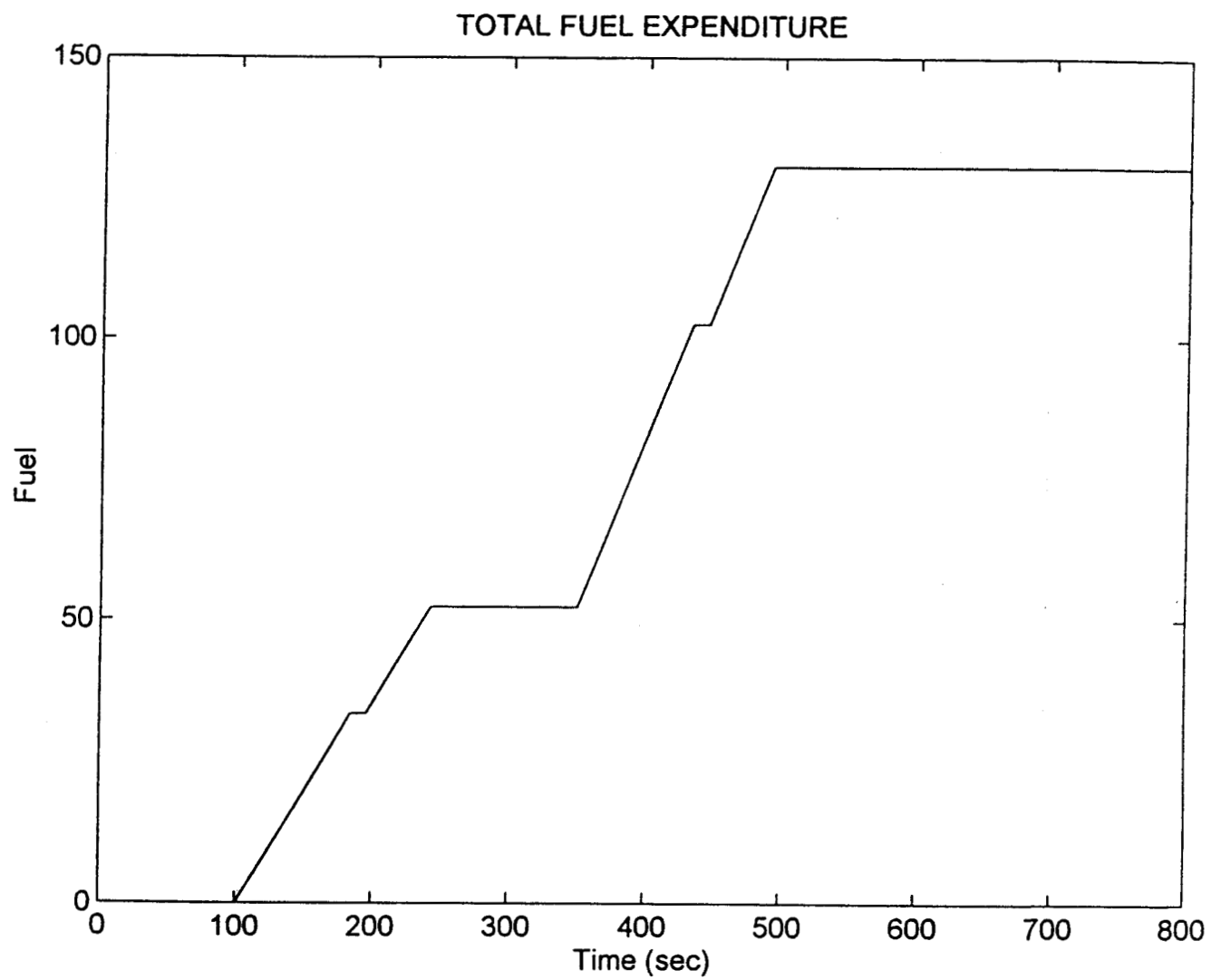
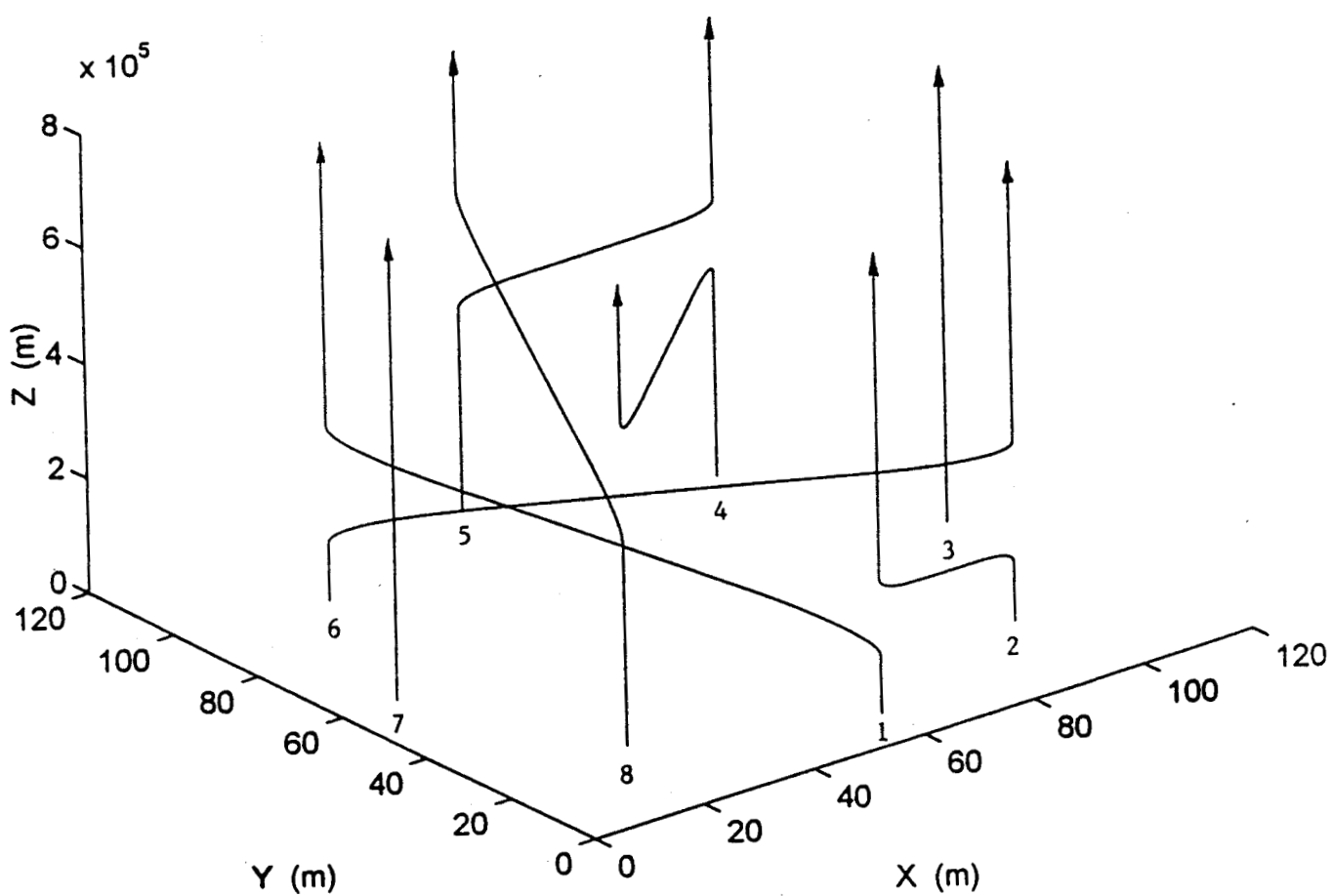Fig.11 Total normalized fuel expenditure during reconfiguration.

SPACECRAFT MOTION IN 3D-SPACE
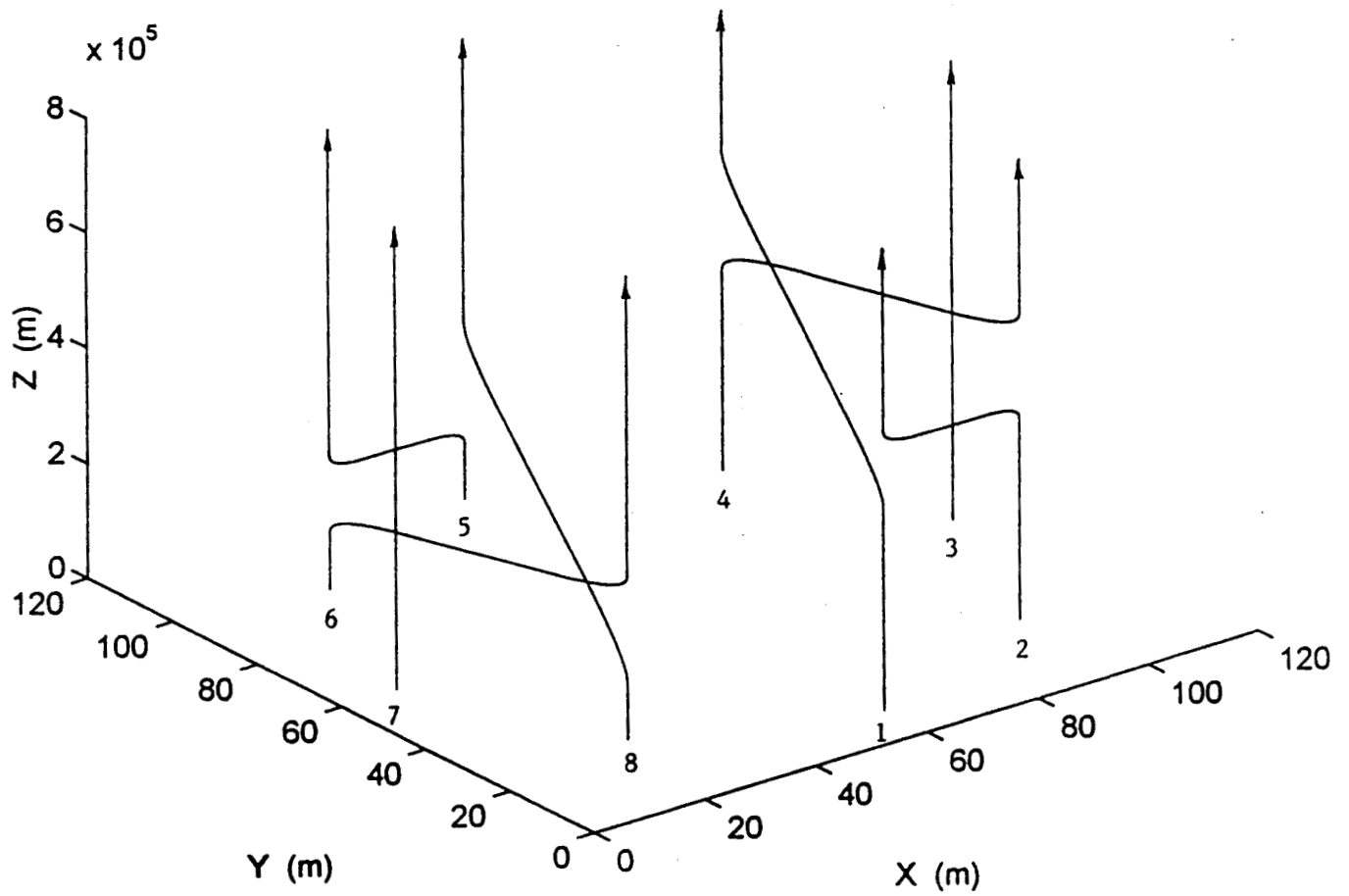
Fig.12 Trajectories of two sets of four identical spacecraft in R^3 during reconfiguration by cycling spacecraft 1,2,4 and 5,6,8.

FUEL FOR SPACECRAFT 1,2,4

Fig.13a Normalized fuel expenditures for spacecraft 1,2 and 4 during reconfiguration.

Fig.13b Normalized fuel expenditures for spacecraft 5,6 and 8 during reconfiguration.
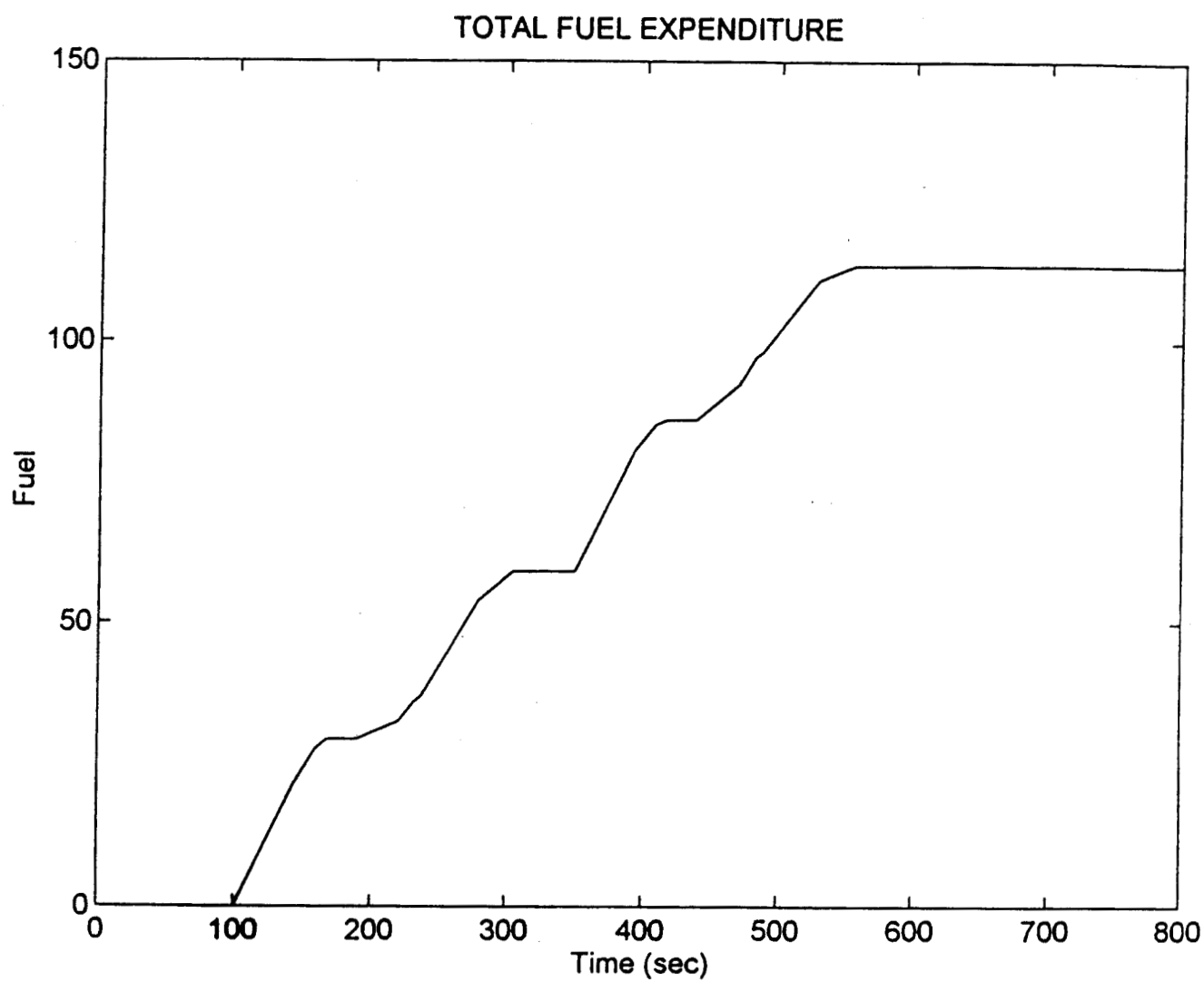
Fig.14 Total normalized fuel expenditure during reconfiguration by cycling spacecraft 1,2,4 and 5,6,8.

SPACECRAFT MOTION IN 3D-SPACE

Fig.15 Trajectories of two sets of four identical spacecraft in R^3 during reconfiguration by cycling spacecraft 1,2,6 and 4,5,8.
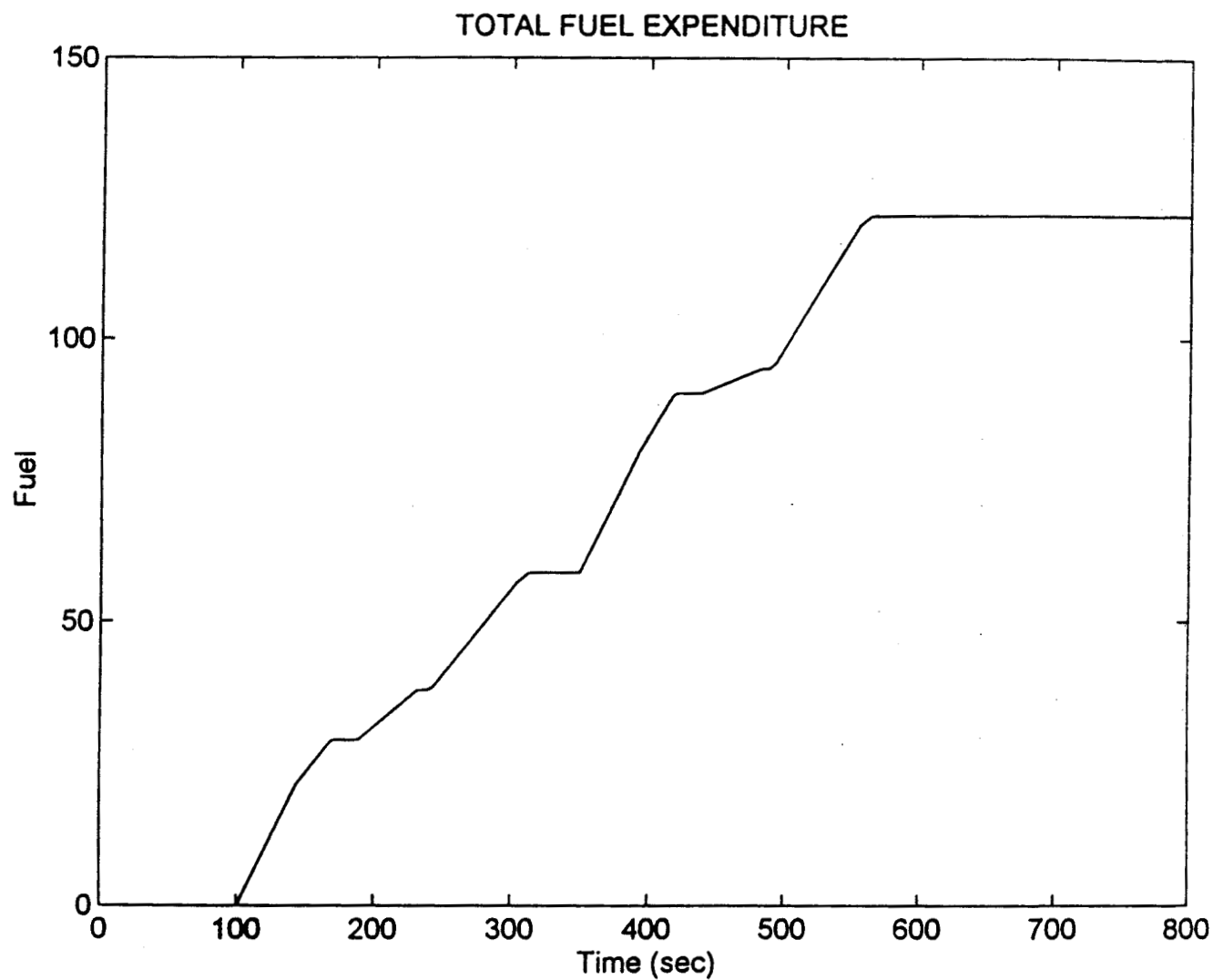
Fig.16 Total normalized fuel expenditure during reconfiguration
by cycling spacecraft 1,2,6 and 4,5,8.
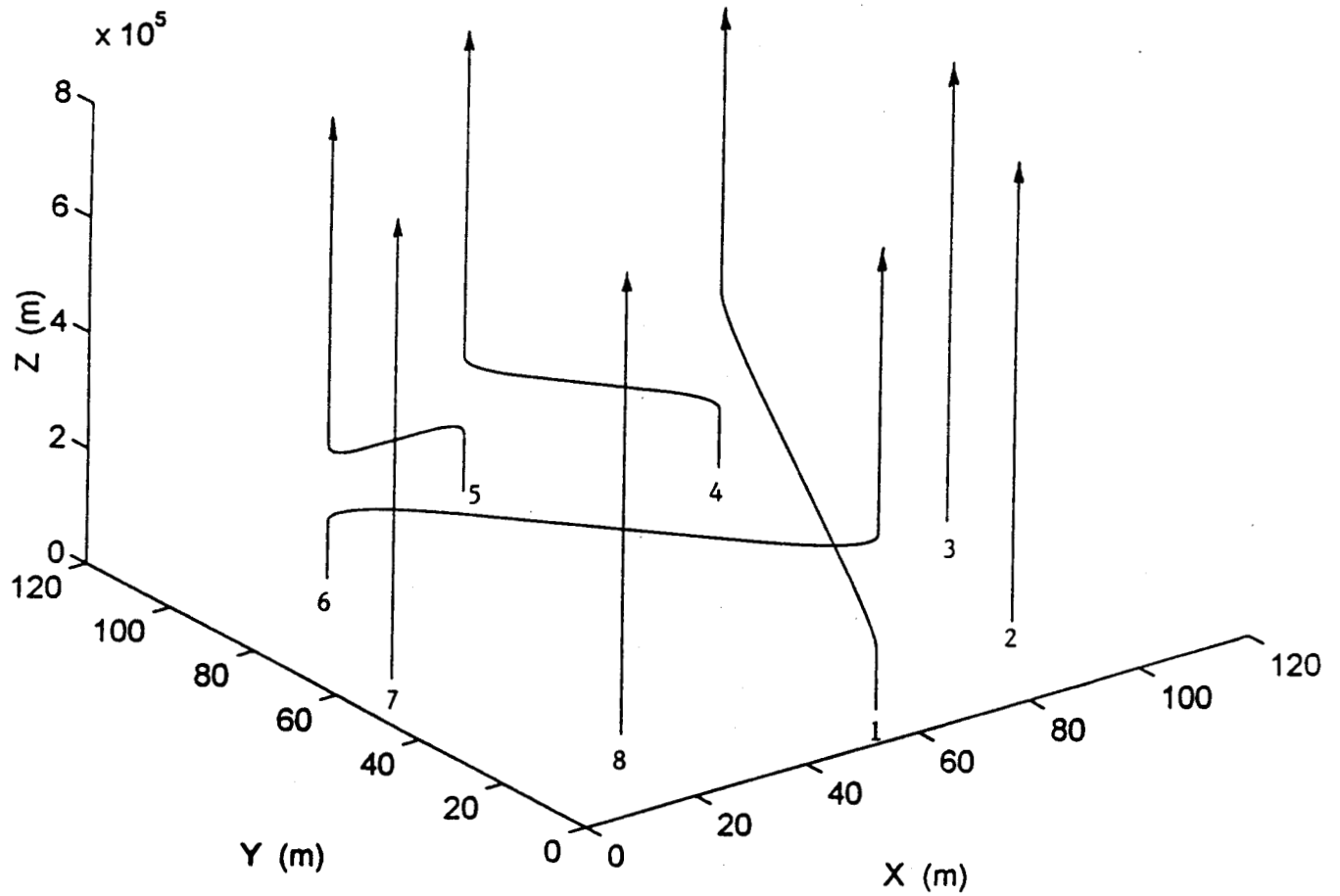
SPACECRAFT MOTION IN 3D-SPACE

Fig.17 Trajectories of two sets of four identical spacecraft in R^3 during reconfiguration by cycling spacecraft 1,4,5 and 6.
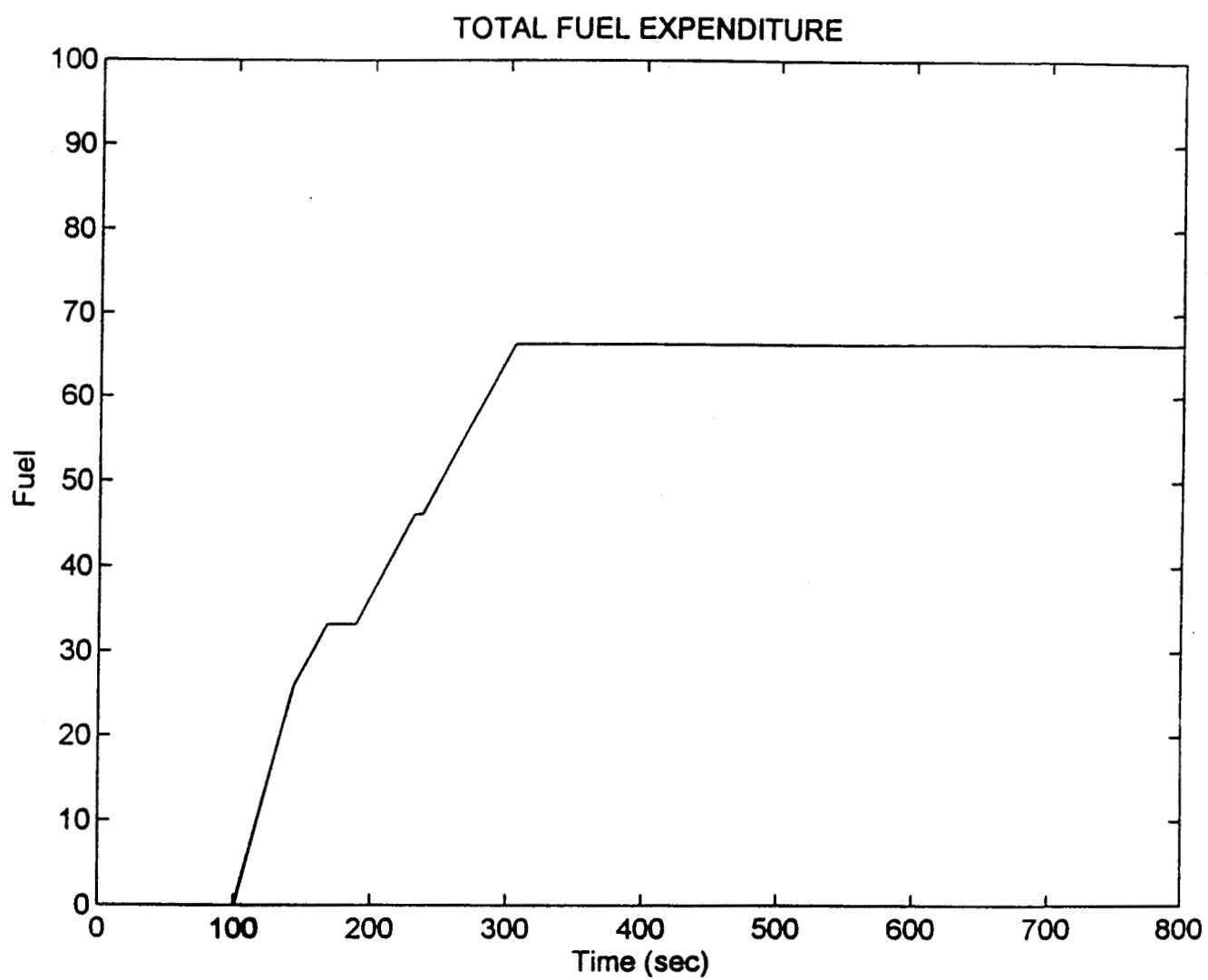
TOTAL FUEL EXPENDITURE

Fig.18 Total normalized fuel expenditure during reconfiguration
by cycling spacecraft 1,4,5, and 6.